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MORE ABOUT THE Q-DEFORMED H-ATOM WAVE FUNCTIONS: NORMAL AND ABNORMAL SERIES

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ABSTRACT

The 3-dim quantum Euclidean space and the q-deformed Schrödinger equation are further investigated. The reality condition is taken into account by introducing the right derivatives as well as the left ones. It seems that from the properties of the q-derivatives, corresponding to a fixed "energy level" there are infinitive numbers of q deformed wave functions. Among them only one belongs to the normal series, its radial wave function (rwf) has the same number of nodes as that of its classical counterpart. All others have more nodes in their rwf's, and then fall into the abnormal series. This is illustrated by solving the q-Schrödinger equation explicitly for the s-wave solutions.

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1 Introduction

In a previous paper [1] (hereafter we refer it as I, and adopt most of notations in I), based on the consistency method for the differential calculi on the quantum planes [2], we developed the differential calculus on the 3–dimensional quantum Euclidean space, wrote down the quantum Schrödinger equation, and gave the concrete form of solutions for the first several "energy levels" for r^{-1} "potential". It seems that the "energy levels" of this deformed H–atom are very similar to the classical one, namely, $E_n = E_1/[n]^2$. The similar equation has also been discussed by others, and the solutions corresponding to the quantum harmonic oscillators were obtained [3]. It is not clear whether the so-called "quantum Schrödinger equation" has anything to do with our real physics or not. But it is still very interesting to clarify this equation and its solutions, especially the H-atom solutions and harmonic oscillator solutions, since these two are the most typical and most important examples in the quantum mechanics.

It is also known that for the quantum planes, the differential calculi possess a lot of new features. How to take into account of the reality condition is one of the open questions. Several authors have considered this question for the quantum Minkowski space [4]. We gave another approach in dealing with the reality condition on the quantum Euclidean space [5]. We will outline this approach below to convince you that the equation given in [1,3] and the solutions discussed there are reliable. We would also like to indicate that the "quantum Schrödinger equation" can be solved by using the series method which is widely used in solving the the ordinary Schrödinger equation. This is illustrated by giving all the s-wave solutions of the H-atom. New features arise from the properties of the derivative operators on the quantum plane. Corresponding to a fixed "energy level", we have got infinite numbers of solutions. In the $q \to 1$ limit, all these solutions coincide to one which is just the solution of the classical solution of Schrödinger equation with the corresponding energy. But for $q \neq 1$, all these solutions appear differently. Among them only one belongs to the principal (or normal) series, its radial wave function (rwf) has the same number of nodes as that of its classical counterpart. All others have more nodes in their rwf's. Then we put them into the abnormal series.

In Sec.2 we briefly describe the approach we used to take into account of the reality condition. Besides the left derivatives one introduced before, the right derivatives are also needed, and they exchange their role under the conjugate. The real "quantum Schrödinger equation" is established in Sec.3, and the solutions of the deformed s-wave H-atom wave functions are obtained. Some discussions will be given in Sec.4.

2 Derivatives and reality

Quantum group $SO_q(3)$ is characterized by an $\check{R}$ matrix which is the solution of the Yang–Baxter equation

$$\check{R}_{12}\,\check{R}_{23}\,\check{R}_{12} = \check{R}_{23}\,\check{R}_{12}\,\check{R}_{23}\,, \tag{2.1}$$

and which has three different eignevalues [6]

$$(\check{R}-\lambda_2)(\check{R}-\lambda_1)(\check{R}-\lambda_0) = 0\,, \tag{2.2}$$

with $\lambda_2(q) = q, \lambda_1(q) = -q^{-1}$ and $\lambda_0(q) = q^{-2}$. These eigenvalues correspond to the quintet, the triplet and singlet, respectively. The projection operators corresponding to these three eigenvalues can be expressed as

$$\begin{aligned} Q^{(2)} &= \frac{(\check{R}-\lambda_0)(\check{R}-\lambda_1)}{(\lambda_2-\lambda_0)(\lambda_2-\lambda_1)} , \\ Q^{(1)} &= \frac{(\check{R}-\lambda_0)(\check{R}-\lambda_2)}{(\lambda_1-\lambda_0)(\lambda_1-\lambda_2)} , \\ Q^{(0)} &= \frac{(\check{R}-\lambda_1)(\check{R}-\lambda_2)}{(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)} , \end{aligned} \tag{2.3}$$

with the properties

$$Q^{(0)} + Q^{(1)} + Q^{(2)} = E, \quad Q^{(\alpha)}\, Q^{(\beta)} = \delta^{\alpha\beta}\, Q^{(\beta)} . \tag{2.4}$$

From (2.1) and (2.3) it is easy to see that

$$Q^{(\alpha)}_{12}\, \check{R}_{23}\, \check{R}_{12} = \check{R}_{23}\, \check{R}_{12}\, Q^{(\alpha)}_{23}, \quad \check{R}_{12}\, \check{R}_{23}\, Q^{(\alpha)}_{12} = Q^{(\alpha)}_{23}\, \check{R}_{12}\, \check{R}_{23} . \tag{2.5}$$

As stated in I the singlet eigenvectors of $\check{R}$ are scaled to give the metric

$$\check{R}^{ij}_{k\ell}\, g(q)^{k\ell} = \lambda_0(q)\, g(q)^{ij}, \quad g(q)_{ij}\, \check{R}^{ij}_{k\ell} = \lambda_0(q)\, g(q)_{k\ell} \tag{2.6}$$

which satisfy the following useful relations

$$g(q)_{ij}\, g(q)^{jk} = \delta^k_i, \quad g(q)^{ij}\, g(q)_{jk} = \delta^i_k \tag{2.7a}$$

$$g(q)_{ij}\, g(q)^{ij} = q + 1 + q^{-1} \tag{2.7b}$$

and

$$g(q)_{ij}\, \check{R}^{jk}_{\ell m} = \check{R}^{-1\ kn}_{i\ell}\, g(q)_{nm}, \quad \check{R}^{ij}_{k\ell}\, g(q)^{\ell m} = g(q)^{in}\, \check{R}^{-1\ jm}_{nk} . \tag{2.7c}$$

The quantum plane is defined [7] in terms of N variables (coordinates) $x^i, i = 1, 2, \ldots N$, which belong to a noncommutative associative C^*–algebra and satisfy the commutation relations

$$x^i\, x^j - B^{ij}_{k\ell}\, x^k\, x^\ell = 0 . \tag{2.8}$$

The (left) derivatives are defined as ususal,

$$\partial_i \equiv \frac{\partial}{\partial x^i}, \qquad (\partial_i\, x^j) = \delta^j_i . \tag{2.9}$$

The derivatives are also noncummutative,

$$\partial_i\, \partial_j = \partial_\ell\, \partial_k\, F^{k\ell}_{ji} , \tag{2.10}$$

and satisfy the commutation relations with the coordinates x

$$\partial_i\, x^j = \delta^j_i + C^{jk}_{i\ell}\, x^\ell\, \partial_k . \tag{2.11}$$

The consistency [2] among (2.8), (2.10) and (2.11) requires [1, 3]

$$C = -\lambda_1^{-1}\, \check{R} = q\check{R}, \qquad B = F = E - Q^{(1)} . \tag{2.12}$$

With the help of the metric g, we can raise the indices of the derivatives as

$$\partial^i = g(q)^{ij}\, \partial_j . \tag{2.13}$$

Then (2.10) and (2.11) can be put into the form

$$\partial^i\, \partial^j = B^{ij}_{k\ell}\, \partial^k\, \partial^\ell \tag{2.10'}$$

$$\partial^i\, x^j = g(q)^{ij} + q^{+1}\, \check{R}^{-1\ ij}_{k\ell}\, x^k\, \partial^\ell . \tag{2.11'}$$

In a similar way we can introduce the right derivatives $\tilde{\partial}_j$,

$$(x^i\, \tilde{\partial}_j) = \delta^i_j . \tag{2.14}$$

The commutation relations among themselves are supposed to be

$$\tilde{\partial}_i\, \tilde{\partial}_j = \tilde{\partial}_k\, \tilde{\partial}_\ell\, \tilde{F}^{\ell k}_{ji} \tag{2.15}$$

and those with coordinates

$$x^i\, \tilde{\partial}_j = \delta^i_j + \tilde{\partial}_k\, x^\ell\, \tilde{C}^{ki}_{\ell j} . \tag{2.16}$$

The consistency sequences $\tilde{C} = C$ and $\tilde{F} = F$. By raising the indices of $\tilde{\partial}$

$$\tilde{\partial}^i = \tilde{\partial}_j\, g(q)^{ji} \tag{2.17}$$

(2.15) and (2.16) can be rewritten as

$$\tilde{\partial}^i\, \tilde{\partial}^j = B^{ij}_{k\ell}\, \tilde{\partial}^k\, \tilde{\partial}^\ell \tag{2.15'}$$

$$x^i\, \tilde{\partial}^j = g(q)^{ij} + q\, \check{R}^{-1\ ij}_{k\ell}\, \tilde{\partial}^k\, x^\ell . \tag{2.16'}$$

The enlarged algebra generated by x, ∂ and $\tilde{\partial}$ is closed by introducing the commutation relations between ∂ and $\tilde{\partial}$, which can easily be fixed from consistency

$$\partial^i\, \tilde{\partial}^j = G^{ij}_{k\ell}\, \tilde{\partial}^k\, \partial^\ell, \quad G = C^{-1} . \tag{2.18}$$

Eqs.(2.8), (2.10′), (2.11′), (2.15′), (2.16′) and (2.18) complete our enlarged algebra.

Now consider the real structure of the quantum Euclidean space. Take the $*$ operator as the generization of the complex conjugate. Parameter q is required to be real $q^* = q$. The conjugate of the coordinates x^i are defined as in the classical case

$$\bar{x}_i \equiv (x^i)^* = x^j\, g(q)_{ji} . \tag{2.19}$$

The conjugation of the derivatives is accordingly defined as

$$(\partial^i)^* = \tilde{\partial}^j\, g(q)_{ji}, \quad (\tilde{\partial}^i)^* = \partial^j\, g(q)_{ji} \tag{2.20}$$

with the change of the acting directions considered.

It is easy to check that the conjugation defined above is an involution, e.g.,

$$\left((x^i)^*\right)^* = x^i, \quad \left((\partial^i)^*\right)^* = \partial^i, \quad \left((\tilde{\partial}^i)^*\right)^* = \tilde{\partial}^i \tag{2.21}$$

and the definition relations for the enlarged algebra are compatible with the conjugation. As a matter of fact, under the conjugation (2.8) and (2.18) are invariant, whereas (2.10′) ↔ (2.15′) and (2.11′) ↔ (2.16′). It is also easy to check the compatibility between the conjugation operation and the quantum group transformation [5].

3 Quantum Schrödinger equation and s–wave H–atom solutions

The quadratic derivative operators which are invariant under the action of $SO_q(3)$ can now be obtained:

$$g(q)_{ij}\ \partial^i\ \partial^j = \partial^i\ \partial_i, \qquad g(q)_{ij}\ \tilde{\partial}^i\ \tilde{\partial}^j = \tilde{\partial}_j\ \tilde{\partial}^j\ , \tag{3.1a}$$

$$g(q)_{ij}\ \partial^i\ \tilde{\partial}^j, \qquad g(q)_{ij}\ \tilde{\partial}^i\ \partial^j = \tilde{\partial}^i\ \partial_i = \tilde{\partial}_j\ \partial^j\ . \tag{3.1b}$$

It can easily be seen from (2.20) that

$$(\partial^i\ \partial_i)^* = (\tilde{\partial}_k\ \tilde{\partial}^k), \quad (\tilde{\partial}_k\ \tilde{\partial}^k)^* = (\partial^i\ \partial_i) \tag{3.2a}$$

while terms in (3.1b) are invariant, e.g.,

$$\begin{aligned}(\tilde{\partial}^i\ \partial_i)^* &= (\partial_i)^*(\tilde{\partial}^i)^* \\ &= g(q)^{ij}\ \tilde{\partial}_j\ \partial^k\ g(q)_{ki} = \tilde{\partial}_k\ \partial^k\end{aligned} \tag{3.2b}$$

and another term is related to this one through (2.18). For the time being, mainly for simplicity, we take the combination

$$\frac{1}{2}\ (\partial^i\ \partial_i + \tilde{\partial}_i\ \tilde{\partial}^i) \tag{3.3}$$

as the q–analogue of the "real" Laplacian operator Δ which is invarinat both under the $SO_q(3)$ and under the conjugation. The meaning is specified as an operator acting upon the "wave functions", i.e.,

$$\Delta\ \psi = \frac{1}{2}\ (\partial^i\ \partial_i)\psi + \frac{1}{2}\ \psi(\tilde{\partial}_i\ \tilde{\partial}^i)\ .$$

Then the quantum Schrödinger equation can be written as

$$(-\Delta + V)\psi = E\ \psi\ . \tag{3.4}$$

As stressed in I, the "length square" $r^2 = g(q)_{ij}\ x^i\ x^j$, which is invariant under the $SO_q(3)$ transformation, is the centre of the algebra generated by the coordinate x. But this is not true when we consider the enlarged algebra generated by x, ∂ and $\tilde{\partial}$. In fact we have, from (2.10) and (2.15),

$$\partial^i\ r^2 = q^{-1}(1+q)x^i + q^2\ r^2\ \partial^i, \quad r^2\ \tilde{\partial}^i = q^{-1}(1+q)x^i + \tilde{\partial}^i\ q^2\ r^2\ . \tag{3.5}$$

More generally,

$$\partial^i\ r^\nu = g^{-1}[[\nu]]r^{\nu-2}\ x^i + q^\nu\ r^\nu\ \partial^i, \quad r^\nu\ \tilde{\partial}^i = q^{-1}[[\nu]]r^{\nu-2}\ x^i + \tilde{\partial}^i\ q^\nu\ r^\nu \tag{3.6}$$

where ν is an integer and $[[\nu]]$ is defined as

$$[[\nu]] = \frac{1-q^\nu}{1-q} \tag{3.7}$$

so that $[[0]] = 0, [[1]] = 1, [[2]] = 1+q, [[3]] = 1+q^1+q^2 \ldots$ and $[[-\nu]] = -q^{-\nu}\ [[\nu]]$. And other addition formulae can be obtained

$$[[m]] + q^m[[n]] = [[m+n]], \qquad q[[n]] + [[n+2]] = [[2]]\ [[n+1]]\ . \tag{3.8}$$

Defining the q–exponential function as

$$\exp_q(z) = \sum_{n=0}^{\infty}\ \frac{z^n}{[[n]]!} \tag{3.9}$$

we can get

$$\partial^i \exp_q\left(-\frac{r}{a}\right) = -q^{-1}\ \frac{x^i}{ar} \exp_q\left(-\frac{r}{a}\right) + \exp_q\left(-\frac{qr}{a}\right) \partial^i \tag{3.10a}$$

$$\exp_q\left(-\frac{r}{a}\right) \tilde{\partial}^i = -q^{-1}\ \frac{x^i}{ar} \exp_q\left(-\frac{r}{a}\right) + \tilde{\partial}^i \exp_q\left(-\frac{qr}{a}\right) \tag{3.10b}$$

and

$$\Delta \exp_q\left(-\frac{r}{a}\right) = -\frac{1+q}{q^2ar} \exp_q\left(-\frac{r}{a}\right) + \frac{1}{a^2} \exp_q\left(-\frac{r}{a}\right)\ . \tag{3.11}$$

In the following disucssion we focus our attention to the s–wave solutions for Eq.(3.4), i.e., the solutions only depend on the variable r. In this case we can show that

$$\begin{aligned}(x^i\ \partial_i)r^n &= q^{-1}[[n]]\ r^n + (qr)^n(x^i\ \partial_i) \\ (\partial^i\ \partial_i)r^n &= q^{-2}[[n]]\ [[n+1]]r^{n-2} + q^{n-1}[[2]]\ [[n]]r^{n-2}(x^i\ \partial_i) + q^{2n}r^n(\partial^i\ \partial_i)\end{aligned} \tag{3.12}$$

and similar for $\tilde{\partial}$. Then we can see that: *acting upon the functions of r only*, we can set

$$\begin{aligned}(x^i\ \partial_i) &= q^{-1}\ r\ D_r, \\ (\partial^i\ \partial_i) &= r^{-2}D_r\ r^2 D_r = q^2 D_r^2 + \frac{[[2]]}{r}\ D_r\end{aligned} \tag{3.13}$$

where the q–difference is defined as

$$D_r\ F(r) = \frac{F(r) - F(qr)}{r - qr} \tag{3.14}$$

and then

$$D_r\ r^n = [[n]]r^{n-1} + (qr)^n\ D_r\ . \tag{3.15}$$

Now for hydrogen–like systems, $\underline{V(r) = -\frac{Ze^2}{r}}$ we seek the solutions corresponding to the s–wave in the form of

$$\psi(r) = \sum_{\nu=0}^{N-1}\ \alpha_\nu \left(-\frac{r}{d}\right)^\nu \exp_q\left(-\frac{r}{d}\right)\ . \tag{3.16}$$

Here instead of $\underline{a}$ used in I, we specify the scale parameter as $\underline{d}$, since in its classical analogue which means the diameter rather than the radius. Setting $\underline{Ze^2 = \lambda/d}$, by a tedious calculation we obtain the recurrence formulae

$$\alpha_\nu\ q^{-2}[[\nu]]\ [[\nu+1]] - \alpha_{\nu-1}\left(\lambda - q^{\nu-3}[[2]]\ [[\nu]]\right) + \alpha_{\nu-2}\left(q^{2(\nu-2)} + Ed^2\right) = 0\ . \tag{3.17}$$

As in the classical case, the series must be terminated to give the eigenvalue of energy E. Suppose the highest coefficient is α_{N-1}, all $\alpha_\nu = 0$ for $\nu \geq N$. Then (3.17) for $\nu = N+1$ reads

$$\alpha_{N-1}\left(q^{2(N-1)} + Ed^2\right) = 0\ . \tag{3.18}$$

This gives the relation between the energy and the scale,

$$E = -q^{2(N-1)}\ d^{-2}\ . \tag{3.19}$$

Different from the classical case, the energy $\underline{E}$ depends on not only the scale of the system $\underline{d}$ but also the terminate point N. Other equations in (3.17), from $\nu = 1$ to N, give a set of N homogeneous equations for N parameters $\alpha_\nu (\nu = 0$ to $N-1)$. The condition for non–zero solution of α_ν demands the vanishing of its coefficient determinant, which gives the eigenvalue of the scale d, and then E according to (3.19).

For $N = 1$, the only non-zero coefficient is α_0. (3.19) turns out to be

$$E = -d^{-2}\ . \tag{3.20}$$

And the only non–trivial equations in (3.17) is $\nu = 1$

$$\alpha_0\left(\lambda - q^{-2}[[2]]\ [[1]]\right) = 0\ .$$

This gives $\lambda = q^{-2}[[2]]$, i.e., $d = d_1 = [[2]]\ q^{-2}(Ze^2)^{-1}$. So that

$$E = -d_1^2 = -q^4 Z^2 e^2/[[2]]^2 = E_1\ . \tag{3.21}$$

For $N = 2$, we have

$$E = -q^2\ d^{-2}$$

from (3.19). Two coefficients α_0, α_1 satisfy the equations

$$\begin{aligned} \alpha_1\ q^{-2}[[1]]\ [[2]] - \alpha_0(\lambda - q^{-2}[[1]]\ [[2]]) &= 0\ , \\ -\alpha_1(\lambda - q^{-1}[[2]]^2) + \alpha_0(1 - q^2) &= 0\ . \end{aligned}$$

The solvable condition is

$$\Delta_2 = (\lambda - q^{-1}[[2]]^2)(\lambda - q^{-2}[[2]]) - (1 - q^2)q^{-2}[[2]] = 0$$

or

$$\left[\lambda - q^{-1}(1+q)(q+q^{-1})\right]\left[\lambda - q^{-1}(1+q)\right] = 0\ . \tag{3.22}$$

Two solutions:

(i)

$$\begin{aligned} \lambda = q^{-1}(1+q)(q+q^{-1}) \qquad &\text{or} \qquad d = d_2 = [2]qd_1 \\ \frac{\alpha_1}{\alpha_0} = q^2, \qquad & \qquad E = E_2 = E_1/[2]^2 \end{aligned} \tag{3.23}$$

(ii)

$$\begin{aligned} \lambda = q^{-1}(1+q) \qquad &\text{or} \qquad d = d_1^{(1)} = qd_1 \\ \frac{\alpha_1}{\alpha_0} = q - 1 \qquad & \qquad E = E_1^{(1)} = E_1 \end{aligned} \tag{3.24}$$

For $N = 3$,

$$E = -q^4\ d^{-2}$$

from (3.19). Three coefficients α_0, α_1 and α_2 satisfy the equations

$$\begin{aligned} -\alpha_2(\lambda - q^0[[2]]\ [[3]]) + \alpha_1\ q^2(1 - q^2) &= 0 \\ \alpha_2\ q^{-2}[[2]]\ [[3]] - \alpha_1(\lambda - q^{-1}[[2]]\ [[2]]) + \alpha_0(q^0 - q^4) &= 0 \\ \alpha_1\ q^{-2}[[1]]\ [[2]] - \alpha_0(\lambda - q^{-2}[[2]]\ [[1]]) &= 0\ . \end{aligned}$$

The solvable condition

$$0 = \Delta_3 = (\lambda - [3]\ [[2]])(\lambda - [2][[2]])(\lambda - [[2]])\ . \tag{3.25}$$

Three solutions:

(i)

$$\begin{aligned} d &= d_3 = [3]\ [[2]](Ze^2)^{-1} = [3]q^2 d_1, E = E_3 = E_1/[3]^2 \\ \alpha_0 &= 1, \quad \alpha_1 = q^2(q^2+1), \quad \alpha_2 = q^6\,\frac{q^2+1}{q^2+q+1}; \end{aligned} \tag{3.26}$$

(ii)

$$\begin{aligned} d &= d_2^{(1)} = [2]\ [[2]](Ze^2)^{-1} = [2]q^2 d_1, E = E_2^{(1)} = E_1/[2]^2 \\ \alpha_0 &= 1, \quad \alpha_1 = q^3 + q - 1, \quad \alpha_2 = q^3(q-1); \end{aligned} \tag{3.27}$$

(iii)

$$\begin{aligned} d &= d_1^{(2)} = [[2]](Ze^2)^{-1}q^2 d_1, E = E_1^{(1)} = E_1 \\ \alpha_0 &= 1, \quad \alpha_1 = q^2 - 1, \quad \alpha_2 = q(q-1)^2\ . \end{aligned} \tag{3.28}$$

Similarly, for $N = 4$

$$E = -q^6 d^{-2}, \quad 0 = \Delta_4 = (\lambda - [4]q[[2]])(\lambda - [3]q[[2]])(\lambda - [2]q[[2]])(\lambda - q[[2]])\ . \tag{3.29}$$

Four solutions:

(i)

$$\begin{aligned} d &= d_4 = [4]q^3 d_1, \quad E = E_1/[4]^2 \\ \alpha_0 &= 1, \quad \alpha_1 = q^2(q^4+q^2+1), \\ \alpha_2 &= q^6(q^2+1)(q^2-q+1), \quad \alpha_3 = q^{12}(q^2-q+1)/(q^2+q+1); \end{aligned} \tag{3.30}$$

(ii)

$$\begin{aligned} d &= d_3^{(1)} = [3]q^3 d_1, \quad E = E_1/[3]^2, \\ \alpha_0 &= 1, \quad \alpha_1 = q^5 + q^3 + q - 1, \\ \alpha_2 &= q^3(q^2+1)(q^5+q^3-1)/(q^2+q+1), \\ \alpha_3 &= q^8(q-1)(q^2+1)/(q^2+q+1); \end{aligned} \tag{3.31}$$

(iii)

$$\begin{aligned} d &= d_2^{(2)} = [2]q^3 d_1, \quad E = E_1/[2]^2 \\ \alpha_0 &= 1, \quad \alpha_1 = q^4 + q^2 - 1 \\ \alpha_2 &= q(q-1)(q^2+1)(q^2+q-1), \quad \alpha_3 = q^5(q-1)^2; \end{aligned} \tag{3.32}$$

(iv)

$$\begin{aligned} d &= d_1^{(3)} = q^3 d_1, \quad E = E_1, \\ \alpha_0 &= 1, \quad \alpha_1 = q^3 - 1, \\ \alpha_2 &= q(q-1)(q^3-1), \quad \alpha_3 = q^3(q^3-1). \end{aligned} \tag{3.33}$$

Among all these ten wave functions, four have energy E_1, three have E_2, two E_3 and one E_4. This procedure can be continued. The general situation can be summarized as follows:

For level $E_1 = -1/d_1^2 = -(q\ e)^4/[[2]]^2$

$$\psi_{100}^{(k)} = \exp_q\left(-\frac{r}{q^k\ d_1}\right)\ \Pi_{\alpha=0}^{(k-1)}\left[1-(q-1)\ \frac{q^\alpha\ r}{q^k\ d_1}\right]\ . \tag{3.34a}$$

For level $E_2 = E_1/[2]^2$

$$\psi_{200}^{(k)} = \left[1 - q^2\ \frac{r}{d_2}\right] \exp_q\left(-\frac{r}{q^k\ d_2}\right)\ \Pi_{\alpha=0}^{(k-1)}\ \left[1-(q-1)\ \frac{q^\alpha\ r}{q^k\ d_2}\right]\ . \tag{3.34b}$$

For level $E_3 = E_1/[3]^2$

$$\psi_{300}^{(k)} = \left[1 - q^2(q^2+1)\frac{r}{d_3} + q^6\frac{q^2+1}{q^2+q+1}\left(\frac{r}{d_3}\right)^2\right] \exp_q\left(-\frac{r}{q^k\ d_3}\right)\ \Pi_{\alpha=0}^{(k-1)}\left[1-(q-1)\ \frac{q^\alpha\ r}{q^k\ d_3}\right]. \tag{3.34c}$$

For level $E_4 = E_1/[4]^2$

$$\psi_{400}^{(k)} = \left[1 - q^2(q^4+q^2+1)\frac{r}{d_4} + q^6(q^2+1)(q^2-q+1)\left(\frac{r}{d_4}\right)^2 - q^{12}\frac{q^2-q+1}{q^2+q+1}\left(\frac{r}{d_4}\right)^3\right] \times$$

$$\exp_q\left(-\frac{r}{q^k\ d_4}\right)\ \Pi_{\alpha=0}^{(k-1)}\left[1-(q-1)\frac{q^\alpha\ r}{q^k\ d_4}\right]\ . \tag{3.34d}$$

Here

$$d_n = [n]q^{n-1}\ d_1 = \frac{[[2n]]}{[[2]]}\ d_1\ .$$

That is to say, for fixed energy level E, we have found infinite numbers of solutions. The solution presented in I corresponding to $k = 0$ all have the same number of nodes in their rwf as for their classical counterparts: $n_r = n - \ell - 1$. These solutions are said to belong to the normal series. All others have more nodes in their rwf, $n_r = N - \ell - 1 = (n+k) - \ell - 1$, with $k = 1$ to infinitive. All these solutions are said to belong to the abnormal series.

For the normal series, $k = 0 \ \ (N = n)$, we can further obtain the coefficients in the expansion (3.16)

$$\alpha_\nu = \frac{1}{[[\nu]]!\ [[\nu+1]]!}\ \frac{[[2(N-1)]]!!}{[[2(N-\nu-1)]]!!}\ \alpha_0\ . \tag{3.35}$$

Then the polynomials in (3.16) are indeed some kind of deformation of the associated Laguerre function. The details of these solutions and the relevant topics will be discussed elsewhere.

4 Brief Discussion

In this paper we only focus our attention to the q–deformation of the s–wave solutions of hydrogen–like atom. There are two possibilities: i) For fixed level, all the functions with different k are equivalent to each other in some sense. Then the further explanation of the deformed wave function must consider this equivalence. ii) Functions with different k are really different. Then we must further clarify the meaning of this new degeneracy in the deformed theory. The same method can be used to discuss the higher wave solutions and even other systems like harmonic oscillators. The appearance of the new degeneracy and the existence of the abnormal series is a new feature, which is closely related to the properties of the q–derivatives. We hope that this can be clarified further in the near future.

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