IC/90/230
INTERNAL REPORT
(Limited Distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

K-HOMOLOGY AND K-COHOMOLOGY CONSTRUCTIONS OF RELATIONS *

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ABSTRACT

One of the important homology [cohomology] theories, based on systems of covering of the space, is the homology [cohomology] theory of relations defined by C.H. Dowker [8]. In the present work, by using the idea of K-homology and K-cohomology groups [1],[5], different varieties of the Dowker's theory are introduced and studied. These constructions are defined on the category of pairs of topological spaces and over a pair of coefficient groups.

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August 1990

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0. INTRODUCTION

One of the widespread cohomology theories applicable in sufficiently wide categories of topological spaces is the Alexandrov-Čech theory [6],[7],[9]. Some other theories are equivalent to Alexandrov-Čech theory as Alexander-Spanier theory [2],[12],[13] and the Vietoris theory [3],[14]. The Alexandrov-Čech homology theory on the category of topological spaces satisfies all axioms of the homology theory except one; namely, the exactness axiom. The lack of this axiom partially explains why advantageous use has been made of the cohomology groups in whole series of problems.

Any relation between the elements of a set X and the elements of a set Y is associated with two simplicial complexes M and L. A simplex of M (of L) is a finite set of element of X (of Y) related to a common element of Y (of X). In particular, the relation of being an element of a set of a covering is a relation between the points of a space and the sets of the coverings of the space. One of the complexes associated with this relation is the nerve of the covering, its K-homology and K-cohomology groups are used in the present work to define the Alexandrov-Čech K-homology and K-cohomology groups of the space. The other associated complex is the Vietoris complex, which has points of the space as its vertices, is used in defining the Vietaris K-homology and Alexander K-cohomology groups. It is proved that the nerve and the Vietoris complexes of any covering of the space have isomorphic K-homology and K-cohomology groups. Therefore, the Alexandrov-Čech K-homology [K-cohomology] and the Vietaries K-homology [Alexander K-cohomology] constructions of relations are isomorphic. Moreover, we show that the Alexandrov-Čech K-cohomology construction of relations, based on the system of all locally-finite open coverings, is a contravariant δ-functor [9].

1. K-HOMOLOGY AND K-COHOMOLOGY GROUPS OF RELATION

In this article some varieties of K-homology and K-cohomology groups are introduced.

Let K be a locally-finite simplicial complex, p a non-negative integer, and (G, G') a pair of commutative groups. Denote by Q the category of the pairs of topological spaces and their continuous maps. Consider that $\Omega(X, A)$ is the directed set, under the refinement relation [9], of all locally-finite open coverings $\alpha = (\alpha_1, \alpha_2)$ of $(X, A) \in Q$. Let $L_\alpha = (L_{\alpha_1}, L_{\alpha_2})$ and $M_\alpha = (M_{\alpha_1}, M_{\alpha_2})$ be the nerve and the Vietoris pair of α, respectively [4],[8]. Note that L_{α_1} is a locally-finite complex but it is not necessary that M_{α_1} to be so.

Consider the group $C_p(L_{\alpha_1})[C^p(L_{\alpha_1})]$ of the p–dimensional K-chains [K-cochains] of L_{α_1} over (G, G'), i.e. an element $x_p[x^p]$ of $C_p(L_{\alpha_1})$ $[C^p(L_{\alpha_1})]$ is a collection $\{x_\tau\}[\{x^\tau\}]$ of $(p + d\tau)$-dimensional chains x_τ [cochains x^τ] of L_{α_1} over G such that the coefficients of $x_\tau[x^\tau]$ belong to G' for almost all simplexes $\tau \in K$, where $d\tau$ denotes the dimension dim τ. Denote by $C_p(L_\alpha)[C^p(L_\alpha)]$ the group of the p–dimensional K-chains [K-cochains] of the pair L_α over

(G, G'), [9]. The boundary and coboundary maps:

$$\partial : C_p(L_\alpha) \to C_{p-1}(L_\alpha) \quad \text{and} \quad \delta : C^p(L_\alpha) \to C^{p+1}(L_\alpha)$$

are defined by

$$(\partial x_p)_\tau = \partial x_\tau + (-1)^{p+d\tau} x_{\partial\tau} \quad \text{and}$$

$$(\delta x^p)^\tau = \delta x^\tau + (-1)^{p+d\tau+1} x^{\delta\tau} ,$$

where

$$x_{\partial\tau} = \sum_{\sigma\in K} [\tau, \sigma] x_\sigma ,\ x^{\delta\tau} = \sum_{\rho \in K} [\rho, \tau] ,$$

$[\tau, \sigma]$ denotes incidence number, $\dim \sigma = dim \tau - 1$, and $\dim \rho = \dim \tau + 1$, [1],[5]. The K-homology [K-cohomology] group $H_p(L_\alpha)[H^p(L_\alpha)]$ of L_α over (G, G') is the homology [cohomology] group of the chain [cochain] complex $\{C_p(L_\alpha), \partial\}[\{C^p(L_\alpha), \delta\}]$.

In order to define the K-homology and K-cohomology groups of the pair M_α, consider the direct system $\{M_{\alpha a} = (M_{\alpha_1 a}, M_{\alpha_2 a}),\ \Lambda(\alpha)\}$ under the inclusion relation, where $M_{\alpha a}$ denotes an arbitrary star-finite subpair of M_α, and $a < b$ in $\Lambda(\alpha)$ if there is an inclusion $\rho^\alpha_{ab} : M_{\alpha a} \subset M_{\alpha b}$. The K-homology [K-cohomology] group of M_α over (G, G') is the direct [inverse] limit group;

$$H_p(M_\alpha) = \overset{\lim}{\longrightarrow}\{H_p(M_{\alpha a}), \rho^\alpha_{ab*}\}$$

$$[H^p(M_\alpha) = \overset{\lim}{\longleftarrow}\{H^p(M_{\alpha a}), \rho^{\alpha *}_{ba}\}] .$$

Let $\alpha < \beta$ in $\Omega(X, A)$. The projection on (X, A) is the relation map $\pi : (X, A, \beta_1, \beta_2, \varepsilon, \varepsilon) \to (X, A, \alpha_1, \alpha_2, \varepsilon, \varepsilon)$ which is the identity on X and maps each $u \in \beta_1$ to an element $v \in \alpha_1$ for which $u \subset v$ and such that whenever $u \in \beta_2$, $v \in \alpha_2$, [8]. This map defines contiguous simplicial maps $\pi_{\beta\alpha} : L_\beta \to L_\alpha$ and $\bar{\pi}_{\beta\alpha} : M_\beta \to M_\alpha$. Then the induced homomorphisms $\pi_{\beta\alpha *} : H_p(L_\beta) \to H_p(L_\alpha)$ and $\bar{\pi}_{\beta\alpha *} : H_p(M_\beta) \to H_p(M_\alpha)$ are uniquely defined. Denote by $S_p(X, A)$ and $S^V_p(X, A)$ the inverse spectrums $\{H_p(L_\alpha), \pi_{\beta\alpha *}\}$ and $\{H_p(M_\alpha), \bar{\pi}_{\beta\alpha *}\}$, respectively. Similarly, we define the direct spectrums $S^P(X, A) = \{H^p(L_\alpha), \pi^*_{\alpha\beta}\}$ and $S^{PV}(X, A) = \{H^p(M_\alpha), \bar{\pi}^*_{\alpha\beta}\}$.

Definition 1.1 The inverse limit groups

$\check{H}_p(X, A; G, G', \Omega) = \overset{\lim}{\leftarrow} S_p(X, A)$ and $\bar{H}_p(X, A; G, G', \Omega) = \overset{\lim}{\leftarrow} S^V_p(X, A)$ are called p-the dimensional Alexandrov-Čech K-homology group and Victories K-group of relations of (X, A) over (G, G') respectively. The direct limit groups

$\check{H}^p(X, A; G, G'; \Omega) = \overset{\lim}{\rightarrow} S^P(X, A)$ and

$\bar{H}^p(X, A; G, G'; \Omega) = \overset{\lim}{\rightarrow} S^{PV}(X, A)$ are called the p-dimensional Alexandrov-Čech K-cohomology group and Alexander K-cohomology group of relations of (X, A) over (G, G') respectively.

In case $A = \phi$ or $X = A$ we obtain the corresponding groups for the space X or A. It can be proved the following result.

Theorem 1.1 If K consists of one vertex and $G' = G$ then the Alexandrov-Čech K-homology [K-cohomology] and the Victoris K-homology [Alexander K-cohomology] groups are isomorphic to the Alexandrov-Čech homology [cohomology] and the Vietoris homology [Alexander cohomology] groups, respectively.

2. INDUCED HOMOMORPHISMS, BOUNDARY AND COBOUNDARY OPERATORS

The induced homomorphisms for the homology and cohomology groups given in Definition 1.1 are defined as follows: Let $f : (X, A) \to (Y, B)$ be in the category Q. Corresponding to f there is at least one relation map $\bar{f} : (X, A; \sigma'_1, \sigma'_2, \varepsilon, \varepsilon) \to (Y, B; \sigma_1, \sigma_2; \varepsilon, \varepsilon)$, [8]. Consider that $f_{\sigma'\sigma} : L_{\sigma'} \to L_\sigma$ and $\bar{f}_{\sigma'\sigma} : M_{\sigma'} \to M_\sigma$ are the associated simplicial maps of $\bar{f}$. Also f defines the function $\phi_f : \Omega(Y, B) \to \Omega(X, A)$ by: $\phi_f(\sigma) = \sigma' = f^{-1}(\sigma)$. The map ϕ_f and the homomorphisms $f_{\sigma'\sigma *}[\bar{f}_{\sigma'\sigma *}]$ form a map $\Phi(f) : S_p(X, A) \to S_p(Y, B)[\bar{\Phi}(f) : S^V_p(X, A) \to S^V_p(Y, B)]$, [9]. Analogously, the maps $\Psi(f) : S^P(Y, B) \to S^p(X, A)[\bar{\Psi}(f) : S^{pV}(Y, B) \to S^{pV}(X, A)]$ are defined.

Definition 2.1 The limit homomorphisms of the maps $\Phi(f), \bar{\phi}(f), \Psi(f)$ and $\bar{\Psi}(f)$ are called f, the induced, by homomorphisms. They are denoted by f_*, $\bar{f}_*$, f^* and $\bar{f}^*$ respectively.

The K-homology and cohomology groups, defined above, of (X, A), X and A are defined as limits of suitable spectrums of groups defined over the directed sets $\Omega(X, A)$, $\Omega(X)$ and $\Omega(A)$ respectively. In order to define the boundary and coboundary operators and discuss the exactness, it will be convenient to have equivalent definitions in which all these spectrums are defined over the same directed set. It appears that the directed set $\Omega(X, A)$ is most suitable for this purpose.

For each $\alpha \in \Omega(X, A)$ the short exact sequence

$$0 \to C_p(L_{\alpha_2}) \xrightarrow{i_\alpha} C_p(L_{\alpha_1}) \xrightarrow{j_\alpha} C_p(L_\alpha) \longrightarrow 0 \ldots \tag{2.1}$$

of chain complexes, defines the exact sequence, [10]:

$$T_\alpha \ldots \to H_p(L_{\alpha_2}) \xrightarrow{i_{\alpha*}} H_p(L_{\alpha_1}) \xrightarrow{j_{\alpha*}} H_p(L_\alpha) \xrightarrow{\partial_\alpha} H_{p-1}(L_{\alpha_2}) \longrightarrow \ldots$$

where ∂_α denotes the connected homomorphism of the sequence (2.1), [11]. T_α is called K-homology sequence of L_α over (G, G'). Similarly, the K-cohomology sequence T^α of L_α and K-homology[K-cohomology] sequence $T^V_\alpha [T^\alpha_V]$ of M_α over (G, G') are defined. If $\alpha < \beta$, then the projection π on (X, A) defines the simplicial maps $\pi_{\beta\alpha}, \bar{\pi}_{\beta\alpha}$, which in turn, induce the homomorphsims $\Pi_{\beta\alpha} : T_\beta \to T_\alpha$, $\Pi_{\alpha\beta} : T^\alpha \to T^\beta$, $\bar{\Pi}_{\beta\alpha} : T^V_\beta \to T^V_\alpha$ and $\bar{\Pi}_{\alpha\beta} : T^\alpha_V \to T^\beta_V$.

Definition 2.2 The limit sequence of the inverse spectrum $\{T_\alpha, \Pi_{\beta\alpha}\}[\{T_\alpha^V, \bar{\Pi}_{\beta\alpha}\}]$:

$$\ldots \to \check{H}_p(A;G,G')_{(X,A)} \xrightarrow{i'_*} \check{H}_p(X;G,G')_{(X,A)} \xrightarrow{f'_*} \check{H}_p(X,A;G,G';\Omega) \xrightarrow{\partial'} \check{H}_{p-1}(A;G,G')_{(X,A)} \to \ldots$$

$$[\ldots \bar{H}_p(A;G,G')_{(X,A)} \xrightarrow{\bar{i}'_*} \bar{H}_p(X;G,G')_{(X,A)} \xrightarrow{*} \bar{H}_p(X,A;G,G';\Omega) \xrightarrow{\bar{\partial}'} \bar{H}_{p-1}(A;G,G')_{(X,A)} \to \ldots]$$

is called the Alexandrov-Čech [Vietoris] adjusted K-homology sequence of (X,A). The Alexandrov-Čech [Alexander] adjusted K-cohomology sequence of (X,A):

$$\ldots \to \check{H}_p(X;A;G,G';\Omega) \xrightarrow{f'^*} \check{H}^p(X;G,G')_{(X,A)} \xrightarrow{i'^*} \check{H}_p(A;G,G')_{(X,A)} \xrightarrow{\delta'} \check{H}^{p+1}(X,A;G,G';\Omega) \to \ldots$$

$$[\ldots \bar{H}_p(X,A;G,G';\Omega) \xrightarrow{\bar{f}'^*} \bar{H}^p(X;G,G')_{(X,A)} \xrightarrow{\bar{i}'^*} \bar{H}_p(A;G,G')_{(X,A)} \xrightarrow{\bar{\delta}'} \bar{H}^{p+1}(X,A;G,G';\Omega)_{(X,A)} \to \ldots]$$

is the limit sequence of the direct spectrum $\{T^\alpha, \Pi_{\alpha\beta}\}[\{T_V^\alpha, \bar{\Pi}_{\alpha\beta}\}]$.

The next result gives a comparison between the groups with the subscript (X,A) with the groups without this subscript.

Lemma 2.1 There are isomorphisms:

$$\check{H}_p(A;G,G';\Omega) \simeq \check{H}_p(A;G,G')_{(X,A)}, \quad \check{H}_p(X;G,G';\Omega) \simeq \check{H}_p(X;G,G')_{(X,A)}$$

$$\check{H}^p(A;G,G')_{(X,A)} \simeq \check{H}^p(A;G,G';\Omega) \quad \underline{\text{and}} \quad \check{H}^p(X;G,G')_{(X,A)} \simeq \check{H}^p(X;G,G');\Omega) .$$

Proof Define the maps $\emptyset : \Omega(X,A) \to \Omega(A)$ and $\psi : \Omega(X,A) \to \Omega(X)$ by: if $\alpha \in \Omega(X,A)$ and $\alpha' = \alpha_2 \cap A$, then $\emptyset_\alpha = \alpha'$ and $\psi\alpha = \alpha_1$. Since $L_{\alpha_2} = L_{\emptyset\alpha}$ and $L_{\alpha_1} = L_{\psi\alpha}$, the maps $\emptyset$ and ψ and the appropriate identity maps of the K-homology groups yield maps of inverse spectrums: $\Phi : S_p(A) \to \{H_p(L_{\alpha_2}), \pi_{\beta_2\alpha_2^*}, \Omega(X,A)\}$ and $\Psi : S_p(X) \to \{H_p(L_{\alpha_1}), \pi_{\beta_1\alpha_1^*}, \Omega(X,A)\}$ where$S_p(A)$ and $S_p(X)$ are defined over $\Omega(A)$ and $\Omega(X)$, respectively. The limits of Φ and Ψ are $\emptyset_\infty : \check{H}^p(A;G,G';\Omega) \to \check{H}_p(A;G,G')_{(X,A)}$ and $\psi_\infty : \check{H}_p(X;G,G';\Omega) \to \check{H}_p(X;G,G')_{(X,A)}$. For cohomology, Φ and Ψ are maps of direct spectrums; their limit are $\emptyset^\infty : \check{H}^p(A;G,G')_{(X,A)} \to \check{H}^p(A;G,G',\Omega)$ and $\psi^\infty : \check{H}^p(X;G,G')_{(X,A)} \to \check{H}^p(X;G,G';\Omega)$. In order to prove that the homomorphsims $\emptyset_\infty, \psi_\infty, \emptyset^\infty$ and $\psi^\emptyset$ are isomorphisms it suffices to mention that the image of $\emptyset$ is a cofinal subset of $\Omega(A)$ and that the image of ψ is a cofinal subset of $\Omega(X)$, [9]. Similarly, we have isomorphisms $\bar{\emptyset}_\infty : \bar{H}_p(A;G,G';\Omega) \to \bar{H}_p(A;G,G')_{(X,A)}$, $\bar{\psi}_\infty : \bar{H}_p(X;G,G';\Omega) \to \bar{H}_p(X;G,G')_{(X,A)}$, $\bar{\emptyset}^\infty : \bar{H}^p(A;G,G')_{(X,A)} \to \bar{H}^p(A;G,G',\Omega)$ and $\bar{\psi}^\infty : \bar{H}^p(X;G,G')_{(X,A)} \to \bar{H}^p(X;G,G';\Omega)$ which prove the lemma.

Definition 2.3 The boundary operators

$$\partial_* : \check{H}_p(X,A;G,G';\Omega) \to \check{H}_{p-1}(A;G,G';\Omega), \bar{\partial}_* : \bar{H}(X,A;G,G';\Omega) \to \bar{H}_{p-1}(A;G,G';\Omega)$$

and the coboundary operators

$$\delta^* : \check{H}^p(A;G,G';\Omega) \to \check{H}^{p+1}(X,A;G,G';\Omega) \text{ and } \bar{\delta}^* : \bar{H}^p(X,A;G,G';\Omega) \to \bar{H}^{p+1}(A;G,G';\Omega)$$

are defined by:

$$\partial_* = \emptyset_\infty^{-1}\partial', \bar{\partial}_* = (\bar{\phi}_\infty)^{-1}\bar{\partial}', \delta^* = \delta'(\emptyset^\infty)^{-1} \text{ and } \bar{\delta}^* = \delta'(\bar{\phi}^\infty)^{-1} ,$$

respectively.

Now, we have the triples of functions $\check{H}_* = \{\check{H}_p, f_*, \partial_*\}$, $\bar{H}_* = \{\bar{H}_p, \bar{f}_*, \bar{\partial}_*\}$, $\check{H}^* = \{\check{H}^p, f^*, \delta^*\}$ and $\bar{H}^* = \{\bar{H}^p, \bar{f}^*, \bar{\delta}^*\}$ where $\check{H}_p, \bar{H}_p[\check{H}^p, \bar{H}^p]$ denote the Alexandrov-Čech, Vietoris homology [Alexandrov-Čech, Alexander cohomology] groups which are defined for every object (X,A) of the category Q, and for each non-negative integer p, the functions f_*, $\bar{f}_*$, f^*, $\bar{f}^*$ are the induced homomorphisms by a continuous map $f : (X,A) \to (Y,B)$ of the category Q, and $\partial_*, \bar{\partial}_*[\delta^*, \bar{\delta}^*]$ are the boundary [coboundary] homomorphisms.

Definition 2.4 The Alexandrov-Čech, Vietoris [Alexandrov-Čech, Alexander] K-homology [K- cohomology] constructions of relations are the triples $\check{H}_*, \bar{H}_*[\check{H}^*, \bar{H}^*]$.

3. ISOMORPHISMS

This article is devoted to study the relation between the homology constructions $\check{H}_*, \bar{H}_*$ and also between the cohomology constructions $\check{H}^*, \bar{H}^*$.

Assume that the vertices of L_{α_1} are ordered so that the vertices of any simplex of L_{α_1} have a simple order. Let $L'_\alpha = (L'_{\alpha_1}, L'_{\alpha_2})$ be the barycentric subdivision of L_α, [13], and $\emptyset_\alpha : L'_\alpha \to L_\alpha$ the simplicial map which maps each vertex of L'_α, that is simplex of L_{α_1}, to its first vertex in the given order [8].

Lemma 3.1 The maps $\emptyset_{\alpha *}$ and $\emptyset_\alpha^*$ are isomorphisms.

Proof It is known that any simplicial complex and its barycentric subdivision have the same homology and cohomology groups, [10]. Hence L_α and L'_α have the same K-homology and K-cohomology groups. Define $\psi_\alpha : L'_\alpha \to M_\alpha$ as follows: if y' is a vertex of L'_{α_1}, choose $\psi_\alpha y' \in X$ so that $\psi_\alpha y' \in y$ for each $y \in y'$ and so that, if y' is a vertex of L'_{α_2}, $\psi_\alpha y' \in A$ and $\psi_\alpha y' \in y$ for each $y \in y'$ [8]. The map ψ_α is a simplicial map, and the homomorphisms $\psi_{\alpha *}$ and ψ_α^* are uniquely determined. One can show that the maps $\omega_\alpha = \psi_{\alpha *}\emptyset_{\alpha *}^{-1}: H_p(L_\alpha) \to H_p(M_\alpha)$ and $\eta_\alpha = \emptyset_\alpha^{*-1}\psi_\alpha^* : H^p(M_\alpha) \to H^p(L_\alpha)$ are isomorphisms, [8].

Lemma 3.2 If $\beta \in \Omega(X,A)$, $\alpha \in \Omega(Y,B)$ and $\xi_{\beta\alpha} : L_\beta \to L_\alpha$, $\bar{\xi}_{\beta\alpha} : M_\beta \to M_\alpha$ are the associated simplicial maps of the map of relations $\xi : (X,A,\beta_1,\beta_2,\varepsilon,\varepsilon) \to (Y,B,\alpha_1,\alpha_2,\varepsilon,\varepsilon)$, then $\bar{\xi}_{\beta\alpha} * \omega_\beta = \omega_\alpha \xi_{\beta\alpha *}$ and $\eta_\beta \bar{\xi}^*_{\alpha\beta} = \xi^*_{\alpha\beta}\eta_\alpha$.

Proof Let $\xi'_{\beta\alpha} : L'_\beta \to L'_\alpha$ be a simplicial map induced by $\xi_{\beta\alpha}$. Then the maps $\bar{\xi}_{\beta\alpha}\psi_\beta$, $\psi_\alpha\xi'_{\beta\alpha}$, $\xi_{\beta\alpha}\emptyset_\beta$ and $\emptyset_{\beta\alpha}\xi'_{\beta\alpha}$ are contiguous, [8]. Hence $\bar{\xi}_{\beta\alpha *}\psi_{\beta *} = \psi_{\alpha *}\xi'_{\beta\alpha *}$ and $\emptyset_{\alpha *}\xi'_{\beta\alpha *} = \xi_{\beta\alpha *}\emptyset_{\beta *}$. Therefore, $\bar{\xi}_{\beta\alpha *}\omega_\beta = \omega_\alpha\xi_{\beta\alpha *}$. Similarly, $\eta_\beta\bar{\xi}^*_{\alpha\beta}\eta_\alpha = \xi^*_{\alpha\beta}\eta_\alpha$. Let $\omega_{\alpha_2} = (\psi_\alpha|L'_{\alpha_2})_*(\emptyset_\alpha|L'_{\alpha_2})_*^{-1}$, $\eta_{\alpha_2} = (\emptyset_\alpha|L'_{\alpha_2})^{*-1}(\psi_\alpha|L'_{\alpha_2})^*$. Thus $\bar{\partial}_\alpha\omega_\alpha = \omega_{\alpha_2}\partial_\alpha$ and $\eta_\alpha\bar{\delta}_\alpha = \delta_\alpha\eta_{\alpha_2}$, where $\partial_\alpha[\bar{\partial}_\alpha]$, $\delta_\alpha[\bar{\delta}_\alpha]$ are the connected homomorphisms, [11].

Theorem 3.1 There are natural isomorphisms:

$$\check{H}_p(X,A;G,G';\Omega) \simeq \bar{H}_p(X,A;G,G';\Omega) \text{ and } \bar{H}^p(X,A;G,G';\Omega) \simeq \check{H}^p(X,A;G,G';\Omega)\,.$$

Proof According to Lemma 3.2 the identity map on $\Omega(X,A)$ and the isomorphisms $\{\omega_\alpha\}$ form a map $\lambda : S_p(X,A) \to S_p^V(X,A)$. It is easy to show that the limit homomorphism λ_* of λ is an isomorphism, [9]. In view of Lemma (3.2), ω_α commutes with $f_{\alpha\sigma*}$ and we have $\tilde{\partial}_\alpha\omega_\alpha = \omega_{\alpha_2}\partial_\alpha$. These commutativities prove the naturality of λ_*. For cohomology the proof is analogous.

Thus we have the following main result.

Theorem 3.2 On the category Q the Alexandrov-Čech K- homology [K-cohomology] construction of relations $\check{H}_*[\check{H}^*]$ and the Vietoris K-homology [Alexander K- cohomology] construction of relations $\bar{H}_*[\bar{H}^*]$ and the Vietoris K-homology [Alexander K- cohomology] construction of relations $\bar{H}_*[\bar{H}^*]$ are isomorphic.

4. AXIOMATIC CHARACTERIZATION

In the present article the Alexandrov-Čech K-homology[K-cohomology] construction $\check{H}_*[\check{H}^*]$ is considered from the point of view of the axioms of Eilenberg-Streenrod for the homology and cohomology theories [9].

It is not difficult to show that the construction $\check{H}_*[\check{H}^*]$ is a covariant [contravariant] functor from the category Q to the category $\mathcal{G}$ of commutative groups and their homomorphisms.

Lemma 4.1 Let $f : (X,A) \to (Y,B)$ be in Q and $f/A : A \to B$ be its restriction, then $\partial_* f_* = (f|A)_*\partial_*$ and $f^*\delta^*(f|A)^*$.

Proof The formula for homology is an immediate consequence of the commutativity relations in the diagram

$$\begin{tikzcd}
\check{H}_p(X,A;G,G';\Omega) \arrow[r, "\partial'"] \arrow[d, "f_*"'] & \check{H}_{p-1}(A;G,G')_{(X,A)} \arrow[d, "(f|A)'_*"] & \check{H}_{p-1}(A;G,G';\Omega) \arrow[l, "\theta_\infty"'] \arrow[d, "(f|A)_*"] \\
\check{H}_p(Y,B;G,G';\Omega) \arrow[r, "\partial'"] & \check{H}_{p-1}(B;G;G,G')_{(Y,B)} & \check{H}_{p-1}(B;G,G';\Omega) \arrow[l, "\theta_\infty"']
\end{tikzcd} \quad , \quad (4.1)$$

where $(f|A)_*$ and $(f|A)'_*$ are defined as the appropriate limit maps. The commutativity of the right square of diagram (4.1) follows from the definitions of θ_∞, $(f|A)_*$ and $(f|A)'_*$. Consider that $\alpha \in \Omega(Y,B)$ and $\theta_f\alpha = \alpha' \in \Omega(X,A)$. The naturality of the connected homomorphisms implies that

$$(f|A)_{\alpha'_2\alpha'_2}\partial_{\alpha'} = \partial_\alpha f_{\alpha'\alpha^*}\,.$$

On considering the idea of limit of inverse systems the commutativity of the left square of diagram (4.1) is obtained. The proof for cohomology is similar.

Theorem 4.1 The K- homology [K- cohomology] sequence of $(X,A) \in Q$ over a pair of coefficient groups $(G.G')$ is isomorphic with the adjusted K- homology [K- cohomology] sequence.

Proof We shall only carry out the proof for homology, the proof for cohomology is quite similar. In view of Lemma (2.1) we need only to verify commutativity relations in the diagram

$$\begin{tikzcd}
 & \check{H}_p(A;G,G';\Omega) \arrow[r, "i_*"] \arrow[dd, "\theta_\infty"] & \check{H}_p(X;G,G';\Omega) \arrow[dr, "j_*"] \arrow[dd, "\psi_\infty"] & \\
\check{H}_{p+1}(X,A;G,G';\Omega) \arrow[ur, "\partial_*"] \arrow[dr, "\partial'"'] & & & \check{H}_p(X,A;G,G';\Omega) \\
 & \check{H}_p(A;G,G')_{(X,A)} \arrow[r, "i'_*"] & \check{H}_p(X;G,G') \arrow[ur, "j'_*"'] &
\end{tikzcd}$$

Commutativity in the left–hand triangle is a direct consequence of the definition of ∂_*. In view of [9] (Chapter VIII, 3.18) one can show that $i'_*\theta_\infty i_* = \psi_\infty i_*$ and $j'_*\psi_\infty = j_*$

Corollary 4.1 If $(C,A) \in Q$, then the K- homology [K- cohomology] sequence of (X,A) is semiexact [exact].

Proof According to theorem (4.1) the exactness of the K-homology and K-cohomology sequences is replaced by the exactness of the adjusted sequences. The adjusted sequences are however limits of spectrums of exact sequences defined over $\Omega(X,A)$. Thus the proof of this corollary is a consequence of [9].

Combining Lemma (4.1) and Corollary (4.1) yields the following main result.

Theorem 4.2 The Alexandrov-Čech cohomology construction of relations $\check{H}^*$ is a contravariant δ-functor on the category Q.

Acknowledgments

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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