A NON-LIE ALGEBRAIC FRAMEWORK AND ITS POSSIBLE MERITS FOR SYMMETRY DESCRIPTIONS

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A non-associative algebraic construction is introduced which bears a relation to a Lie algebra \$ \mathfrak{g} \$ paralleling the relation between an associative enveloping algebra and \$ \mathfrak{g} \$. The key ingredient of this algebraic construction is the presence of two parameters which relate it to the enveloping algebra of \$ \mathfrak{g} \$. The analogue of the Poincaré-Birkhoff-Witt theorem is proved for the new algebra. Possibilities of physical relevance are also considered. It is noted that, if fully developed, the mathematical framework suggested by this new algebra should be non-Lie. Subsequently, a certain scheme resulting from specific considerations connected with this (non-Lie) algebraic structure is found to bear striking resemblance to a recent phenomenological theory proposed for explaining CP violation by the K system. Some relevant speculations are also made in view of certain recent trends of thought in elementary particle physics. Finally, in an appendix, a Gell-Mann-Okubo-like mass formula for the new algebra is derived for an SU(3) octet.

I. INTRODUCTION

The effectiveness of Lie algebras in theoretical physics must be considered, by now, as established. Nevertheless, with the deepening of our understanding of problems like symmetry breaking, discrete space-time symmetry violations etc., the need for a more effective algebraic structure emerges as a plausible alternative. Of course, if one were to adopt such an attitude one should not lose sight of the vast and numerous successes of Lie algebras (and groups) in connection with elementary particles. In this regard, one may recall the so-called Jordan algebras introduced as far back as 1934 by Jordan, von Neumann and Wigner \(^{1,2}\) defined by the identities

\[
ab = ba, \quad (\lambda^2)ab = a^2(ba),
\]

where the algebraic product has been denoted by simply placing two elements next to each other. A perhaps more familiar representation of the Jordan product is \(\{a, b\}\).\(^{2}\) Despite several attempts, the question of possible physical usefulness, either of Jordan algebras or various generalizations of them, remains open at this time with no clear-cut application surfacing so far. Accordingly, it makes more sense that any alternative algebraic structure put forth for improving the status quo should have a Lie content in some way or other.

In this respect, the Santilli algebras \(^{3}\), proposed in more recent times, seem to be more promising. The important aspect of the Santilli algebras is Lie admissibility \(^{4}\), a concept which goes back to Albert \(^{5}\). In the present paper we adopt the Santilli version of Lie admissibility to introduce an algebraic structure whose construction rests on a Lie algebra \$ \mathfrak{g} \$ in a way paralleling the construction of the universal enveloping algebra of \$ \mathfrak{g} \$. The added feature is the presence of two parameters \(\lambda, \mu\) which constitute an integral part of the "enveloping" algebra we shall construct. We call this algebraic structure the universal enveloping mutation algebra (UEMA) of \$ \mathfrak{g} \$. The characterization "mutation" has already been introduced by Santilli \(^{3}\) in connection with his proposed model of an algebraic realization of his Lie admissibility axioms. Its meaning should become evident in our case too – after we have introduced the UEMA of a Lie algebra.

The first realization of the adopted Lie admissibility axioms has been given by Santilli and Soliani.\(^{6}\) It can be viewed as a combination of Lie and Jordan algebras with two parameters entering. These parameters express the mixing between the two algebras. Explicitly, consider an associative algebra \$ A \$ and
whose product we denote by placing two elements next to each other. It is well known that the associative product can be used to define the Lie algebra $A_0$ of $A$. Thus, $A_0$ has the same elements as $A$ but the product in the former is specified by

$$[ab] = ab - ba.$$  \hspace{1cm} (1.2)

The Santilli-Soliani algebra $A(\lambda,\mu)$ is introduced by considering a different generalization of $A$. In particular, let $A(\lambda,\mu)$ be the same vector space as $A$ but with the product defined via

$$[ab] = \lambda ab + \mu ba.$$  \hspace{1cm} (1.3)

The algebra $A(\lambda,\mu)$ so defined has been called the $(\lambda,\mu)$ mutation of $A$, mutation algebra for short. Obviously, the word mutation characterizes the flexibility one now has due to the adjustability of parameters $\lambda$ and $\mu$. Note that (1.3) can be rewritten as

$$[ab] = \frac{\lambda - \mu}{2} (ab - ba) + \frac{\lambda + \mu}{2} (ab + ba),$$  \hspace{1cm} (1.4)

which illustrates the mixing of the Lie and Jordan algebra products. One sees right away that a-algebra $A(\lambda,\mu)$ satisfies an "asymptotic condition" in the sense that it reduces directly into a Lie algebra (i.e. $A_0$) as $\lambda \to +1$, $\mu \to -1$. It can be shown very easily that $A(\lambda,\mu)$ is Lie admissible. The transition from a Lie algebra $A$ to the more general (Lie admissible) algebraic structure does not entail, of course, the complete abandonment of the Lie framework. We can, in fact, say quite generally (i.e. irrespective of the particular replacement $A_0 \to A(\lambda,\mu)$) that the replacement of $A$ with any Lie admissible algebra $U$ simply implies the embedding of the Lie into the new framework - or the other way round - with $U$ being identified with $U_0$. Another interesting feature of the Santilli algebras is that they admit a general analytic formulation whose bracket coincides with product (1.4). Thus, they might be attractive as a methodological tool for investigating interpolating fields.

Some initial applications of algebras $A(\lambda,\mu)$ have been investigated in problems such as $SU(3)$ symmetry breaking and mass formulae in $A(\lambda,\mu)$ (with $h(\lambda,\mu)SU(3)$), classical dissipative systems, plasma instabilities and quantum mechanical interpolating fields.

In our construction we start from a Lie algebra $\mathcal{L}$. In particular, our original product is the Lie product, i.e. we have no associative algebra to begin with. It is well known that one can construct at least one associative algebra from $\mathcal{L}$, namely the universal enveloping algebra $\mathfrak{U}(\mathcal{L})$ of $\mathcal{L}$. This algebra could now be used to define a Lie algebra $\mathfrak{U}_L$ or a mutation algebra $\mathfrak{A}(\lambda,\mu)$ via relations (1.2) and (1.3) respectively. A common feature of $\mathfrak{A}, \mathfrak{U}_L$ and $\mathfrak{A}(\lambda,\mu)$ is that they are identical as vector spaces; only their products are different. We also recall (8) that $\mathcal{L}$ is homomorphic, but not isomorphic, to $\mathfrak{A}_L$; i.e. not every element of $\mathfrak{A}_L$ corresponds to an element of $\mathcal{L}$.

Our path for insertions $(\lambda,\mu)$ generalization at the enveloping algebra level of $\mathcal{L}$ will be different from the one we have just sketched. To be somewhat more explicit, the present $(\lambda,\mu)$ generalization is introduced at the level of the tensor product which plays a vital role in the construction of enveloping algebras. In particular, we shall introduce

$$\circ \cdot \circ = \lambda a \otimes b + \mu b \otimes a,$$  \hspace{1cm} (1.5)

Now, (1.5) does not actually give the UEMA product itself. In fact, the UEMA of $\mathcal{L}$ (to be denoted by $\mathfrak{U}(\mathcal{L},\mu)$) is formed as a quotient space of the tensor algebra whose product is $\circ \cdot \circ$. The point is that (1.5) helps turn $\mathcal{L}$ into a Lie admissible algebra in Santilli's sense. Thus, we presume that we have here a new realization of Santilli's identities (1) for a Lie admissible algebra. As will be clearly seen after its construction, our realization is not identical to the universal enveloping algebra $\mathfrak{A}$ of $\mathcal{L}$ as a vector space. In this regard, $\mathfrak{U}(\mathcal{L},\mu)$ constitutes a true non-associative enveloping algebra of a Lie algebra. Conceivably, this property might represent a significant contribution for possible physical applications of algebras defined by the aforementioned Lie admissibility identities. Furthermore, while algebras $A(\lambda,\mu)$ were investigated for the limit $\lambda \to +1$, $\mu \to -1$, we now lead to focus our attention on the other significant limit $\lambda \to -1, \mu \to +1$, where $\mathcal{A}$ is the (associative) enveloping algebra of $\mathcal{L}$.

We shall devote Sec. II to the construction of a UEMA of a Lie algebra as well as the derivation of some of its properties. One central part of this section will be the proof of the analogue to the powerful Poineare-Birkhoff-Witt theorem for a UEMA known, until now, only for a universal enveloping algebra of a Lie algebra.
free parameters one has in possession. For example, if one was to give up the idea of Lie groups as forming a certain nucleus for physical descriptions, the possibility opens that a “Lie” asymmetry is some other (non-Lie) group’s (or algebra’s) symmetry. In this sense, the \( \mathcal{U}(\lambda,\mu) \) could be used, e.g., to write invariant (from the \( \mathcal{U}(\lambda,\mu) \) point of view) Lagrangian expressions which exhibit Lie symmetrical terms parametrized through \( \lambda \) and \( \mu \). It must be admitted, however, that such an alternative is far from obvious at this point since one lacks the parallel of the Lie group that goes with a UEMA. After all, it is the representations of the group which really matter in constructing invariant Lagrangian expressions. We shall deal with such problems in Sec. III. In particular, we shall exhibit a possible connection between a certain \( \mathcal{U}(\lambda,\mu) \) invariance scheme and a superweak phenomenological Lagrangian formulation suggested recently by Hsu to account for CP violation by the \( K \) system.

Needless to emphasize that our understanding of questions revolving around physical applications is still rudimentary. Much more work is necessary before one can talk with confidence about the importance of Lie algebraic generalizations of the sort we are presently introducing. Some speculations along lines concerning relevance to physics are made in Sec. IV.

In the appendix we present another possible application of a UEMA along more orthodox lines. By restricting \( \mathcal{L} \) to \( SU(3) \) we subsequently derive, within the framework of its UEMA, a Gell-Mann-Okubo-like mass formula which contains the parameters \( \lambda \) and \( \mu \) in a way similar, but not equivalent, to the previous derivation in the framework of algebras \( \mathcal{A}(\lambda,\mu) \).

II. UNIVERSAL ENVELOPING MUTATION ALGEBRA OF A LIE ALGEBRA

1) Construction and elementary properties

Throughout this section and the rest of the paper, the term “algebra” stands for associative algebra with an identity element. At times, however, we shall use the adjective “associative” for emphasis. On the other hand, a non-associative algebra should read as “not necessarily associative”. Obviously, we are committed to use the adjective “non-associative” whenever we are referring to a not necessarily associative algebra which lacks any other kind of characterization. A similar comment holds for the “Lie” characterization of an algebra—whenever it applies.

Suppose we are given an algebra \( \mathcal{A} \) whose product is formally denoted by placing two elements of \( \mathcal{A} \) next to each other. We can always form the Lie algebra \( \mathcal{A}_L \) of \( \mathcal{A} \) by introducing the Lie product \( [a,b] = ab - ba \), \( a,b \in \mathcal{A} \) or \( \mathcal{A}_L \). From now on the Lie algebra of an (associative) algebra will be denoted by the subscript \( L \).

Next, let us recall the definition of the universal enveloping (associative) algebra of a Lie algebra \( \mathcal{L} \).

Definition: Let \( \mathcal{L} \) be a Lie algebra. The universal enveloping algebra of \( \mathcal{L} \) is a pair \( (\mathcal{L},\alpha) \) where \( \mathcal{L} \) is an algebra and \( \alpha \) is a homomorphism of \( \mathcal{L} \) into any associative algebra \( \mathcal{A} \) such that if \( \beta \) is any algebra and \( \beta \) is a homomorphism of \( \mathcal{L} \) into \( \mathcal{A} \), then there exists a unique homomorphism \( \delta \) of \( \mathcal{A} \) into \( \mathcal{B} \) such that \( \delta = \beta \). In other words, the existence of a unique homomorphism \( \delta \) in the following diagram:

\[
\begin{array}{ccc}
\mathcal{L} & \longrightarrow & \mathcal{A} \\
\alpha \downarrow & & \delta \\
\mathcal{B} & \rightarrow \ & \mathcal{A}
\end{array}
\]

is the basic requirement for defining the universal enveloping algebra of \( \mathcal{L} \).

We are interested in finding ways by which we can extend the above observations, including the definition, to alternative algebraic structures. For example, the fact that an associative algebra can be turned into a Lie algebra is not an exclusive feature of the former. Accordingly, it is possible that a non-associative algebra \( \mathcal{U} \) with a product denoted by \( a*b \) (\( a,b \in \mathcal{U} \)) gives rise to a Lie algebra \( \mathcal{U}_L \) through the specification

\[
[a,b] \equiv axb - bx a \quad . \quad (2.1)
\]

The crucial property needing verification in each particular case is the Jacobi identity. Indeed, not every non-associative algebra \( \mathcal{U} \) can give rise to a Lie algebra \( \mathcal{U}_L \) through (2.1) because the Jacobi identity cannot be satisfied in general. Any algebra, associative or not, which does give rise to a Lie algebra through (2.1) is called Lie admissible. Our programme is, basically, to generalize the above definition so that it would involve non-associative, Lie admissible algebras. Furthermore, we are interested in obtaining from the new algebraic structures, in some limit, associative algebras which we can interpret as the algebras entering the definition of the universal (associative) enveloping algebra of \( \mathcal{L} \).
As a first attempt, suppose we replace $A$ and $B$ in the above definition by the $(\lambda, \mu)$ mutation algebras $G(\lambda, \mu)$ and $G(\lambda, \mu)$. It is easy to see that no isomorphisms $\alpha^\prime$ and $\beta^\prime$ can be established to replace $\alpha$ and $\beta$ above. In fact, one can show by explicit calculation that the Jacobi identity which is satisfied in $L$ does not, in general, go to zero in $G(\lambda, \mu)$ (or $G(\lambda, \mu)$) unless $\lambda = -\mu = 1$.

A second attempt could have been to replace, in the above definition, $A$, $B$, $\alpha$, and $\beta$ by $G(\lambda, \mu)$ and $G(\lambda, \mu)$ with the simultaneous replacement $A = G(\lambda, \mu)$ and $B = G(\lambda, \mu)$. Clearly, since $G(\lambda, \mu)$ and $G(\lambda, \mu)$ are Lie algebras, it follows that $G(\lambda, \mu)$ and $G(\lambda, \mu)$ are Lie algebras. Such a generalization could possibly have its own merits. We shall not follow it, however, for two reasons which are not necessarily independent. First, $G(\lambda, \mu)$ cannot be constructed independently by itself, i.e., it presupposes the existence (construction) of $A$. Second, the difference between the Lie product in $G(\lambda, \mu)$ and the Lie product in $G(\lambda, \mu)$ is only a factor of $(\lambda - \mu)$. Accordingly, we do not find that such a replacement offers interesting enough motivation for considering it.

In our search for a 2-parameter generalization of the enveloping algebra of $L$ we have no a priori knowledge as to the kind of algebraic structures that are to replace $A$ and $B$. Instead, we shall proceed to the construction of such an algebra for $L$, using the corresponding construction of the universal enveloping algebra as a prototype. Indeed, a universal enveloping algebra can be specified directly in terms of its construction. This procedure is perfectly general since it follows from our definition that any two universal enveloping algebras of $L$ must be isomorphic. 11)

Recall the construction of a universal (associative) enveloping algebra for a Lie algebra $L$. One first forms the (associative) tensor algebra of $L$,

$$\mathcal{T}(L) = \mathbb{F} \oplus L \otimes L \otimes L \otimes \cdots ,$$  

where $\mathbb{F}$ is the field of scalars over which $L$ is defined as a vector space. Next, one considers the ideal $\mathcal{R}$ spanned by all elements of the form

$$[l_1, l_2] = l_1 \otimes l_2 + l_2 \otimes l_1 ; l_1, l_2 \in L ,$$  

and $[l_1, l_2]$ denotes the Lie product. Finally, the algebra $\mathcal{U}$ formed by the quotient $\mathcal{T}(L) / \mathcal{R}$ can be shown (or, alternatively, can be defined) to be the (associative) universal enveloping algebra of $L$.

Suppose now that we form a different kind of a tensor algebra $\mathcal{T}^\prime(L)$, which is not even associative, by the replacement $\otimes = \circ$ where

$$\lambda \cdot \lambda' = \lambda \cdot \lambda_1 + \mu \cdot \lambda_2 , \quad \lambda, \lambda', \mu \in \mathbb{F}.$$  

Explicitly,

$$\mathcal{T}^\prime(L) = \mathbb{F} \oplus L \oplus L \oplus L \oplus \cdots .$$  

As mentioned before, $\mathcal{T}^\prime(L)$ is not associative. One can easily verify that fact by observing, e.g., that $(l_1 \otimes l_2) \circ l_3$ is not necessarily identical with $l_1 \circ (l_2 \otimes l_3) ; l_1, l_2 \in L ; j = 1, 2, 3$. Furthermore, it follows from (2.4) that $\mathcal{T}^\prime(L)$ is a subspace of $\mathcal{T}(L)$.

Consider now the subspace $\mathcal{R}'$ of $\mathcal{T}^\prime(L)$ which is spanned by all elements of the form

$$[l_1, l_2] - l_1 \circ l_2 + l_2 \circ l_1, \quad l_1, l_2 \in L .$$  

$\mathcal{R}'$ is an ideal in $\mathcal{T}^\prime(L)$ under the $\circ$ product. Indeed, consider the mapping $\mathcal{T}^\prime(L) \rightarrow \mathcal{R}$ according to which $L \rightarrow A$ and $\mathcal{R} \rightarrow \mathcal{R}'$. Then $A \otimes A + A' \otimes A'$ where the difference between $A$ and $A'$ is the following. If $A = a_1 \circ \cdots \circ a_n ; l_1, l_2 \in L ; j = 1, \ldots , n$, then

$$A' = a_1 \circ \cdots \circ a_n .$$

But since $A$ is an ideal in $\mathcal{T}^\prime(L)$, then $A \otimes A \in \mathcal{R}$. It follows that $\mathcal{R}' \subseteq \mathcal{R}$ and, therefore (since for every $A'$ in $\mathcal{T}^\prime(L)$ there exists its pre-image $A$ in $\mathcal{T}(L)$), $\mathcal{R}'$ is an ideal of $\mathcal{T}^\prime(L)$.

We can now form the quotient

$$\mathcal{U}(\lambda, \mu) = \mathcal{T}^\prime(L) / \mathcal{R}' .$$  

$\mathcal{U}(\lambda, \mu)$ is a non-associative algebra. Indeed, let $A, B, C \in \mathcal{U}(\lambda, \mu)$.

Explicitly, $A = a + \mathcal{R}'$ , $B = b + \mathcal{R}'$ , $C = c + \mathcal{R}'$ , $a, b, c \in \mathcal{T}(L)$. Denote the product in $\mathcal{U}(\lambda, \mu)$ by $\cdot$. Now

$$(A \cdot B) \cdot C = (a \cdot b + \mathcal{R}') \cdot c + \mu \cdot \lambda (\lambda_1 \circ \cdot \cdot \cdot \circ \lambda_n + \mathcal{R}') .$$  

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whereas

\[ A \times (B \times C) = \lambda^2 a \otimes b \otimes c + \mu \lambda (a \otimes c \otimes b + b \otimes c \otimes a) + \\
+ \mu^2 c \otimes b \otimes a + R'. \]  

(2.9)

The fact that \( \mathcal{U}(\lambda, \mu) \) is non-associative means that when we write, e.g., \( A \times B \times C \) we do not have a uniquely defined element. Since both \( (A \times B) \times C \) and \( A \times (B \times C) \) belong to \( \mathcal{U}(\lambda, \mu) \), the expression \( A \times B \times C \) will stand for either one of the above two elements whenever no confusion can arise. Later on, however, we shall find it necessary to be specific as to how an expression like, e.g., \( A \times B \times C \) is meant to be organized. It also follows from its definition that \( \mathcal{U}(\lambda, \mu) \) is a subset of \( \mathcal{A} \); we can think of \( \mathcal{U}(\lambda, \mu) \) as the restriction of \( \mathcal{A} \) to a hyperplane parametrized by \( \lambda \) and \( \mu \).

We next prove the following.

**Lemma 2.1:** \( \mathcal{U}(\lambda, \mu) \) is Lie admissible.

**Proof:** We form \( \mathcal{U}_L(\lambda, \mu) \) from \( \mathcal{U}(\lambda, \mu) \) as follows. The elements of \( \mathcal{U}_L(\lambda, \mu) \) are the same as those of \( \mathcal{U}(\lambda, \mu) \). The operation in \( \mathcal{U}_L(\lambda, \mu) \) is denoted by \([A, B]\) and is specified by \([A, B] = A \otimes B - B \otimes A\).

It is evident that \( \mathcal{U}_L(\lambda, \mu) \) is a vector space over \( F \) with the Lie product \([,] \) being distributive over addition and \( \alpha[A, B] = \left\{ (\alpha A), B \right\} = \left\{ A, (\alpha B) \right\}, \alpha \in F \), so that, to begin with, \( \mathcal{U}_L(\lambda, \mu) \) is an algebra.

Now, \( A = a + \mathcal{G}' \), \( B = b + \mathcal{G}' \), \( a, b \in \mathbb{T}(\mathbb{L}) \). It follows,

\[ A \times B = a \otimes b + \mathcal{G}' = \lambda^2 (a \otimes b) + \mu (a \otimes b) + \mu (b \otimes a) + \mathcal{G}' \]  

and

\[ B \times A = \lambda (a \otimes b) + \mu (a \otimes b) + \mathcal{G}' \].  

(2.11)

So that

\[ [A, B] = (\lambda - \mu) (a \otimes b - b \otimes a) + \mathcal{G}' \].  

(2.12)

One can now verify the Jacobi identity explicitly or recognize from (2.12) that one basically has a Lie product definition arising from an associative algebra. (Recall that \( \mathbb{T}(\mathbb{L}) \), with multiplication \( \otimes \), is associative.) It is also trivially evident that \([A, A] = 0\) since \( \mathcal{G}' \) is the zero element of \( \mathcal{U}_L(\lambda, \mu) \).

Note that if \( \mathbf{l}_1 , \mathbf{l}_2 \in \mathbf{L} \), then

\[ [\mathbf{l}_1 , \mathbf{l}_2 ] = (\lambda - \mu) (\mathbf{l}_1 \otimes \mathbf{l}_2 - \mathbf{l}_2 \otimes \mathbf{l}_1) + \mathcal{G}' \].  

(2.13)

is the corresponding product in \( \mathcal{U}_L(\lambda, \mu) \). There exists, now, a homomorphism between \( \mathcal{L} \) and \( \mathcal{U}_L(\lambda, \mu) \) according to which \([\mathbf{l}_1 , \mathbf{l}_2 ] \) goes over to \([\mathbf{l}_1 , \mathbf{l}_2 ] \). Indeed, it is not difficult to establish that the homomorphism between \( \mathcal{L} \) and the Lie algebra \( \mathcal{A} \) of the universal enveloping algebra of \( \mathcal{L} \).

The latter is a homomorphism which takes \([\mathbf{l}_1 , \mathbf{l}_2 ] \) to \( \mathbf{l}_1 \otimes \mathbf{l}_2 - \mathbf{l}_2 \otimes \mathbf{l}_1 + \mathcal{G}' \), where \( \mathcal{G}' \), we recall, is an ideal in \( \mathbb{T}(\mathbb{L}) \). Similar arguments would hold for (2.13). In particular, it is very simple to show that, in our case of interest, the Jacobi identity \([[[\mathbf{l}_1 , \mathbf{l}_2 ] , \mathbf{l}_2 ] , \mathbf{l}_1 ] + [[[\mathbf{l}_2 , \mathbf{l}_1 ] , \mathbf{l}_1 ] , \mathbf{l}_2 ] = 0 \) goes into the zero element (i.e., \( \mathcal{G}' \)) of \( \mathcal{U}(\lambda, \mu) \).

The Lie admissible algebra \( \mathcal{U}(\lambda, \mu) \) so constructed will be called by us a universal enveloping mutation algebra (UEMA) for \( \mathcal{L} \). The underlying word "mutation" is that one can think of \( \mathcal{U}(\lambda, \mu) \) as changing according to the parameters \( \lambda \) and \( \mu \). In particular, if \( \mathcal{A} \) is the universal enveloping algebra for \( \mathcal{L} \), it follows that:

\[ \mathcal{U}(\lambda, \mu) \xrightarrow{\lambda \rightarrow \lambda + \mu \rightarrow \mu \rightarrow 0} \mathcal{A}, \mathcal{A}' \].  

(2.14)

2) The analogue of the Poincaré-Birkhoff-Witt theorem for a UEMA

The Poincaré-Birkhoff-Witt theorem (P-B-W for brevity) is of great significance in connection with universal enveloping algebras. From the point of view of physics, it constitutes the underlying mathematical result on which the derivation of the well-known Gell-Mann-Okubo mass formula rests.

The motivations behind this theorem are related to the desire to order the tensor product expansions of \( \mathbb{T}(\mathbb{L}) \). For example, let \( e_1 , \ldots , e_n \) be an ordered basis for \( \mathcal{L} \) and consider \( \mathcal{L} \otimes \mathcal{L} \). Of course, both \( e_1 \otimes e_2 \) and \( e_2 \otimes e_1 \) belong to \( \mathcal{L} \otimes \mathcal{L} \). On the other hand, when one views \( \mathcal{L} \otimes \mathcal{L} \) as part of the tensor algebra \( \mathbb{T}(\mathcal{L}) \), one wants to have available a basis for \( \mathcal{L} \otimes \mathcal{L} \) possessing a certain order. The desired ordering hinges on the way in which the \( e_j \)'s enter the basis elements \( e_j \otimes e_k \) (\( j, k = 1, \ldots , n \) of \( \mathcal{L} \otimes \mathcal{L} \). Suppose, in particular, that we want the

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indices to increase on the right only, i.e. the basis for \( L \otimes L \) to be constituted by elements of the form \( e_j \otimes e_k \) with \( j \leq k \) exclusively. One must ascertain that the elements of the form \( e_j \otimes e_k \) with \( k < j \) are also covered by such a basis.

We call standard any monomial \( e_{j_1} \otimes \cdots \otimes e_{j_k} \) \((\text{degree } k)\) for which the indices strictly appear in an increasing order from right to left, i.e. \( j_s \leq j_r \) for \( s < r \). For any monomial, a parameter called index is introduced which is zero if the monomial is standard. The index has a value \( i \) equal to the minimum number of permutations between successive \( e_j \)'s necessary before the monomial is brought into a standard form. For example, \( \text{ind}(e_2 \otimes e_1) = 1 \), \( \text{ind}(e_3 \otimes e_2 \otimes e_1) = 3 \), \( \text{ind}(e_1 \otimes e_3 \otimes e_1) = 1 \), etc. It is of particular interest to notice that if \( j_r > j_{r+1} \) then

\[
\text{ind}(e_{j_1} \otimes \cdots \otimes e_{j_{r+1}}) = \text{ind}(e_{j_1} \otimes \cdots \otimes e_{j_r} \otimes e_{j_{r+1}}) + 1,
\]

i.e., the permutation of two adjacent \( e_j \)'s changes the index of the monomial by \( \pm 1 \) if \( j_r < j_{r+1} \) \((\text{according to whether } j_r > j_{r+1} \text{ or } j_r < j_{r+1})\). Note that if \( j_{r+1} = j_r \), a permutation causes no change of the index.

The next thing one should realize, still within the context of \( \mathcal{T}(\mathfrak{g}) \), is that

\[
e_{j_2} \otimes e_{j_1} - e_{j_1} \otimes e_{j_2} = [e_{j_3} e_{j_{r+1}}] \text{ mod } R,
\]

(2.16)

It follows that

\[
e_{j_1} \otimes \cdots \otimes e_{j_k} = e_{j_1} \otimes \cdots \otimes e_{j_{r+1}} \otimes e_{j_r} \otimes \cdots \otimes e_{j_k} + [e_{j_3} e_{j_{r+1}}] \text{ mod } R,
\]

(2.17)

In other words, a given monomial can be expressed as a sum of a monomial of the same degree with two members of the product exchanged and a monomial of 1 degree less \((\text{mod } \mathfrak{g})\). This procedure can be applied successively in order to express a monomial with index \( i > 0 \) in terms of a standard monomial of the same degree plus a number of monomials each one of 1 degree less. The latter monomials, if not standard, can again be brought into standard form plus monomials of 2 degrees less than that of the original. In conclusion, every element of \( \mathcal{T}(\mathfrak{g}) \) can be expressed as an \( \mathfrak{g} \)-linear combination of 1 and standard monomials \((\text{mod } \mathfrak{g})\).

The final thing needed to extend the above results to the universal enveloping algebra \( \mathcal{A} \) is a proof for the existence of a mapping \( \sigma : \mathcal{T}(\mathfrak{g}) \to \mathcal{A} \) such that \( \sigma(1) = 1 \), \( \sigma(e_1 \otimes \cdots \otimes e_k) = e_1 \otimes \cdots \otimes e_k \) if \( j_1 \leq j_2 \leq \cdots \leq j_k \) and \( \sigma(e_{j_1} \otimes \cdots \otimes e_{j_k} - e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes e_{j_{r+1}} \otimes \cdots \otimes e_{j_k} = \sigma(e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes e_{j_{r+1}} \otimes \cdots \otimes e_{j_k} \otimes \cdots \otimes e_{j_{r+1}}) \) \((\text{We have denoted, as usual, elements of } \mathcal{A} \text{ by capital letters and the product in } \mathcal{A} \text{ by placing two of its elements next to each other.})\). The existence of this mapping \( \sigma \) opens the way to the PBW theorem which asserts that the cosets of 1 and of standard monomials form a basis for \( \mathcal{A} = \mathcal{T}(\mathfrak{g})/R \).

This powerful theorem cannot be generalized to \( \mathcal{U}(\lambda, \mu) \) in a straightforward manner because the latter is non-associative. We are no longer in a position to liberally make exchanges of two members of a \( \mathcal{U}(\lambda, \mu) \) monomial. For example, if \( j_s > j_{s+1} \), the monomial

\[
(\cdots((e_1 \otimes e_2) \otimes e_3) \cdots) \otimes e_s \otimes e_{s+1} \cdots) \otimes e_k
\]

(2.18)
cannot be expressed, in a manner analogous to (2.17), as

\[
(\cdots((e_1 \otimes e_2) \otimes e_3) \cdots) \otimes e_s \otimes e_{s+1} \cdots) \otimes e_k' + (\cdots(e_1 \otimes e_2) \cdots) \otimes e_{s+1} \cdots) \otimes e_k' \text{ mod } R'.
\]

(2.19)

Before we proceed any further, we must decide what should constitute a standard monomial with respect to the operation \( \otimes' \) \((\text{i.e. in } \mathcal{T}(\mathfrak{g}))\). Since we have no a priori clues, let us be quite general and declare any monomial in \( \mathcal{T}(\mathfrak{g}) \) standard if the ordering \( j_1 \leq j_2 \leq \cdots \leq j_k \) is respected, no matter how the association is made. For example, both \( e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \) and \( e_{j_1} \otimes (e_{j_2} \otimes e_{j_3}) \) are standard with respect to \( \otimes' \), provided \( j_1 \leq j_2 \leq j_3 \). Let us subsequently drop the qualification "with respect to
for simplicity, unless it is needed for clarification purposes.

Getting back to (2.18), we realize that the exchange between \( e_{j+1} \) and \( e_j \) could have taken place along the lines of (2.17) if, instead, we had \( e_{j+1} \)

\[
e_{j+1} \ldots \otimes (e_{j+1} \otimes e_j) \ldots \otimes e_k
\]

(2.20)

where, we omitted all other brackets (associations) in order to stress the fact that what is important in exchanging \( e_{j+1} \) and \( e_j \) is their being directly multiplied by each other. Accordingly, we are naturally led to the following question:

What is the difference between

\[
(a' \otimes e_j) \otimes e_j \otimes b' \quad \text{and} \quad (a' \otimes e_j \otimes e_j) \otimes b',
\]

where \( a' , b' \in \mathcal{T}(\mathcal{A}) \)?

We deal with this question in proving the following.

**Lemma 2.2:** The difference between the two \( k \)-degree monomials in \( \mathcal{T}(\mathcal{A}) \)

\[
(a' \otimes e_j) \otimes e_j \otimes b' \quad \text{and} \quad (a' \otimes e_j \otimes e_j) \otimes b',
\]

the notation already explained) is a sum of \((k-2)\)-degree monomials in \( \mathcal{T}(\mathcal{A}) \) mod \( \mathcal{R} \).

**Proof:** We have previously evaluated:

\[
N_1 \equiv \left( (a' \otimes e_j) \otimes e_j \otimes b' \right) \equiv \left[ \left( a' \otimes e_j \otimes e_j \right) + \lambda \mu \left( e_j \otimes e_j \otimes e_j \otimes e_j \right) \right] \otimes b'
\]

and

\[
N_2 \equiv \left( (a' \otimes e_j \otimes e_j) \otimes b' \right) \equiv \left[ \left( a' \otimes e_j \otimes e_j \otimes e_j \right) + \lambda \mu \left( e_j \otimes e_j \otimes e_j \otimes e_j \right) \right] \otimes b'.
\]

(2.21)

Therefore,

\[
N_1 - N_2 = \lambda \mu \left[ e_j \otimes e_j \otimes e_j \otimes e_j \right] \otimes b'
\]

(2.22)

Finally, the first term on the right-hand side can be brought back to the original format \( (a' \otimes e_j \otimes e_j) \otimes b' \) at the expense of introducing additional

\[
\text{Consider the term } \left( a' \otimes e_j \otimes e_j \otimes e_j \otimes e_j \right) \otimes b'.
\]

Now, \( a' \) has an expansion in \( \mathcal{T}(\mathcal{A}) \) since the product \( \otimes ' \) is related to \( \otimes \). We can now move \( e_{j+1} \) through \( a' \) to the right as already suggested by (2.17). Each time we move \( e_{j+1} \) one position to the right we obtain, in addition, a monomial of a degree less than the degree of the \( \mathcal{T}(\mathcal{A}) \) monomials in the expansion of \( \otimes ' \). When finally, \( e_{j+1} \) is brought all the way to the right, it will cancel with \( e_j \otimes e_j \otimes e_j \) and all that will remain will be a sum of \( \mathcal{T}(\mathcal{A}) \) monomials (mod \( \mathcal{R} \)) of degree \( t-1 \), where \( t \) is the degree of \( a' \).

Denoting this sum by the generic symbol \( b' \) (degree of \( c \equiv t-1 \)), we now have

\[
N_1 - N_2 = \lambda \mu \left[ c \otimes e_j \otimes e_j \otimes e_j \right] \otimes b'.
\]

(2.25)

Repeating the same process with \( e_j \otimes c \) we end up (mod \( \mathcal{R} \)) with a sum of a series of monomials of degree \( t-2 \). Denoting this sum by \( d \) we have

\[
N_1 - N_2 = \lambda \mu \left[ c \otimes e_j \otimes e_j \otimes e_j \right] \otimes b'.
\]

(2.26)

Now \( b' \) also has an expansion in \( \mathcal{T}(\mathcal{A}) \) in terms of monomials of degree \( k-t \). Consequently, \( N_1 - N_2 \) is a sum of (not necessarily standard) monomials of degree \( k-2 \) in \( \mathcal{T}(\mathcal{A}) \) (mod \( \mathcal{R} \)).

Similar arguments hold for the difference \( a' \otimes e_j \otimes e_j \otimes e_j \otimes e_j \).

When we bring a \( \mathcal{T}(\mathcal{A}) \) monomial into the desirable form, we can then apply an argument analogous to that of (2.17). Consider, again, the \( k \)-degree monomial \( (a' \otimes e_j \otimes e_j \otimes e_j \otimes e_j) \otimes b' \) for which \( j_j \) is true. At the expense of introducing a number of \( k-2 \)-degree \( \mathcal{T}(\mathcal{A}) \) monomials mod \( \mathcal{R} \)

we can work with \( (a' \otimes e_j \otimes e_j \otimes e_j \otimes e_j) \otimes b' \). It follows that

\[
(a' \otimes e_j \otimes e_j \otimes e_j \otimes e_j) \otimes b' = (a' \otimes (e_j \otimes e_j \otimes e_j \otimes e_j)) \otimes b' + (e_j \otimes (e_j \otimes e_j \otimes e_j \otimes e_j)) \otimes b' \mod \mathcal{R}'.
\]

(2.27)

Finally, the first term on the right-hand side can be brought back to the original format \( (a' \otimes e_j \otimes e_j \otimes e_j \otimes e_j) \otimes b' \) at the expense of introducing additional
(k-2)-degree monomials in $T(\mathcal{J}) \bmod \mathfrak{M}$. The above arguments can be formalized into the following.

**Lemma 2.3:** A k-degree monomial in $T(\mathcal{J})$ can be expressed (mod $\mathfrak{M}$) as an $F$-linear combination of 1 and standard $T(\mathcal{J})$ monomials of degree $\leq k$ plus an $F$-linear combination of 1 and standard $T(\mathcal{J})$ monomials (mod $\mathfrak{M}$) of degree $\leq (k-2)$.

We have, therefore, found a way to associate together any two members of a given $T(\mathcal{J})$ monomial, we wish to exchange. This is the crucial result necessary for our present non-associative algebraic entity before the analogue of the F-B-W theorem can be extended to $U(\lambda, \mu)$.

We are, accordingly, in a position to follow the steps taken for the proof of the theorem in the case of the enveloping algebra. Unfortunately, we are forced to carry, in addition to monomials in $T(\mathcal{J}) \bmod \mathcal{K}$, monomials in $T(\mathcal{J}) \bmod \mathfrak{N}$ (of a maximum degree smaller by 2 than the corresponding maximum degree of the $T(\mathcal{J})$ monomials).

Consider now, an arbitrary element $a = a' + \mathcal{K}$ of $U(\lambda, \mu)$ where $a' \in T(\mathcal{J})$. We have seen that $a'$ has a decomposition of the form

$$a' = (a'_k + \mathfrak{M}) + (a_{k-2} + \mathfrak{M}),$$

(2.28)

where $a'_k$ is a sum of $T(\mathcal{J})$ standard monomials with maximal degree $k$ and $a_{k-2}$ is a sum of $T(\mathcal{J})$ standard monomials of maximal degree $k-2$. We are, thereby, led naturally to our main result which constitutes a generalization of the F-B-W theorem to $U(\lambda, \mu)$.

**Theorem 2.1:** An arbitrary element of $U(\lambda, \mu) = T(\mathcal{J})/\mathfrak{K}$ has an $F$-linear expansion in terms of $\mathfrak{M}$ cosets of 1 and standard $T(\mathcal{J})$ monomials plus $\mathfrak{N}$ cosets of 1 and standard $T(\mathcal{J})$ monomials. The maximal degree of the latter is 2 less than the maximal degree of the former (monomials).

This theorem is not, obviously, as strong as the corresponding one for universal enveloping algebras. In particular, for a k-degree $U(\lambda, \mu)$ monomial (multiplication in $U(\lambda, \mu)$ denoted by $\cdot$) the theorem asserts

$$a_k \cdot a_k \cdot \ldots \cdot a_k = \sum_{i=1}^{\mathfrak{M}} \alpha_j \cdot a_{j_k} \cdot \ldots \cdot a_{j_k} + \sum_{i=1}^{\mathfrak{M}} \beta_k \cdot a_{j_k} \cdot \ldots \cdot a_{j_k},$$

(2.29)

where $j_1 \leq j_2 \leq \ldots \leq j_\mathfrak{M}$ and $i_1 \leq i_2 \leq \ldots \leq i_\mathfrak{M}$. In (2.29) $\mathfrak{M}$ denotes multiplication in $\mathfrak{M} = T(\mathcal{J})/\mathfrak{K}$ $(A, B \in \mathfrak{M}$ and $a_{j_k} \cdot \mathfrak{M} \in \mathfrak{M}$). Finally, we mention that the particular manner in which the association is made in $a_k \cdot \ldots \cdot a_k$ is irrelevant, given our definition of a $T(\mathcal{J})$ standard monomial. It is possible, however, to sharpen our definition in view of the fact that any kind of association in the above monomial can be brought, e.g., to a form of the form $\ldots ((a'_1 \cdot a'_2) \cdot \ldots) \cdot a'_k$ plus a number of $T(\mathcal{J})$ monomials of degree $\leq (k-2)$ mod $\mathfrak{M}$.

As a specific example, consider the expansion of the expression $(e_1 \cdot e_2) \cdot e_3 + \mathcal{K}$ in terms of standard monomials. One, obtains

$$\begin{align*}
(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} &= \{(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} \} + \lambda \mu \{(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} \} \\
&= (e_1 \cdot e_2 \cdot e_3 + \mathcal{K}) + \lambda \mu \{(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} \} \\
&= (e_1 \cdot e_2 \cdot e_3 + \mathcal{K}) + \lambda \mu \{(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} \} \\
&= (e_1 \cdot e_2 \cdot e_3 + \mathcal{K}) + \lambda \mu \{(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} \} + \mathcal{K} \\
&= (e_1 \cdot e_2 \cdot e_3 + \mathcal{K}) + \lambda \mu \{(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} \} + \mathcal{K} \\
&= (e_1 \cdot e_2 \cdot e_3 + \mathcal{K}) + \lambda \mu \{(e_1 \cdot e_2) \cdot e_3 + \mathcal{K} \} + \mathcal{K}.
\end{align*}$$

(2.30)

where we have literally applied the results contained in Lemma 2.2 as well as Eq. (2.27). Dropping $\mathfrak{K}$ and $\mathfrak{M}$, so we can talk about $U(\lambda, \mu)$ and $\mathfrak{K}$ products, we have:

$$\begin{align*}
(E_1 \cdot E_2) \cdot E_3 &= (E_1 \cdot E_2) \cdot E_3 + \mathcal{K} + E_1 \cdot (E_2 \cdot E_3) + \mathcal{K} + E_2 \cdot (E_1 \cdot E_3) + \mathcal{K} \\
&= (E_1 \cdot E_2) \cdot E_3 + \mathcal{K} + E_1 \cdot (E_2 \cdot E_3) + \mathcal{K} + E_2 \cdot (E_1 \cdot E_3) + \mathcal{K}.
\end{align*}$$

(2.31)

The above result is not actually final; some more (trivial) work is needed, i.e., we must expand the Lie brackets in (2.31) in terms of the Lie basis $\{e_1, e_2, \ldots, e_n\}$ and record rearrangements, according to the procedures already established, wherever necessary.

The weakness of the above theorem may be disturbing at first sight. On the other hand, it is possible that Theorem 2.1 could endow one with enough freedom to proceed towards finding "unconventional" physical applications. Thus, the fact that an arbitrary element of our non-associative algebra $U(\lambda, \mu)$ can be put in a form which contains a standard non-associative and a standard associative part may prove of relevance, especially when one considers representations of $U(\lambda, \mu)$.

For the applications we have in mind it becomes necessary to possess an associative algebra $A_{\lambda, \mu}$ into which one can map elements of $U(\lambda, \mu)$. This
becomes particularly important when one considers representations of $\mathcal{U}(\lambda, \mu)$ which act as transformations on a vector space. Hopefully, this point will be elucidated by the examples in Sec. III as well as that in the appendix. In particular, given the $\mathcal{U}(\lambda, \mu)$ product $A \times B$ (*and* $A, B$ may themselves be monomials with more than one factor), we want to be in a position to write

$$A \times B = \lambda A'B' + \mu B'A', \quad (2.32)$$

where $A', B'$ are the images of $A, B$ in $\mathcal{A}_{\mu}$, and where we have denoted the (associative) product in $\mathcal{A}_{\mu}$ by placing two of its elements next to each other. As we shall argue later, $\mathcal{A}_{\mu}$ can be viewed as similar to the universal enveloping algebra of $\mathfrak{g}$.

More generally, $\mathcal{A}_{\mu}$ can be thought of as an (associative) algebra of $\mathfrak{g}$ monomials. We shall have further comments to make about $\mathcal{A}_{\mu}$ later on.

The feasibility of expression (2.32) can be shown by the following:

**Theorem 2.2:** There exists a mapping $\sigma: \mathcal{U}(\lambda, \mu) \rightarrow \mathcal{A}_{\mu}$, where $\mathcal{A}_{\mu}$ is an associative algebra of $\mathfrak{g}$ monomials, such that:

$$\sigma(A \times B) = \lambda \sigma(A) \sigma(B) + \mu \sigma(A) \sigma(B) \quad (2.33)$$

**Proof:** Consider the mapping $\rho: \mathcal{L} \rightarrow \mathcal{A}_{\mu}$ such that

$$\rho(a \odot b) = \frac{\rho(a) \rho(b)}{(\lambda \cdot \mu)} \quad (2.34)$$

It follows that

$$\rho \left( \left( \ell_{1}, \ell_{2} \right) - \ell_{1} \odot \ell_{2} + \ell_{2} \odot \ell_{1} \right) =$$

$$= \rho \left( \left( \ell_{1} \odot \ell_{2} - \left( \lambda \cdot \mu \right) \ell_{1} \odot \ell_{2} + \left( \lambda \cdot \mu \right) \ell_{2} \odot \ell_{1} \right) \right)$$

$$= \rho \left( \left( \ell_{1}, \ell_{2} \right) - \rho \left( \ell_{1} \right) \rho \left( \ell_{2} \right) + \rho \left( \ell_{2} \right) \rho \left( \ell_{1} \right) \right). \quad (2.35)$$

As we shall remark in more detail in the next section, whenever we are restricting our considerations on the Lie algebra of a Lie group, we can always write

$$\rho \left( \ell_{1}, \ell_{2} \right) = \rho \left( \ell_{1} \right) \rho \left( \ell_{2} \right) - \rho \left( \ell_{2} \right) \rho \left( \ell_{1} \right) \quad (2.36)$$

It then follows that

$$\rho \left( \ell_{1}, \ell_{2} \right) - \ell_{1} \odot \ell_{2} + \ell_{2} \odot \ell_{1} = 0 \quad (2.37)$$

Consequently,

$$\rho(A \times B) = \frac{\lambda}{\lambda \cdot \mu} \rho(A) \rho(B) + \frac{\mu}{\lambda \cdot \mu} \rho(A) \rho(B) \quad (2.38)$$

since, if $\rho$ vanishes on the generator of $\mathfrak{g}$, it vanishes on $\mathfrak{g}$ itself. Furthermore, in (2.38) we have adopted the notation $A = \alpha \odot \beta$, $B = \beta \odot \beta'$. Eq. (2.38) almost guarantees that our mapping has been constructed. To arrive at the special form (2.33), which is particularly useful for the applications we have in mind, we define the mapping $\sigma$ on a $\mathcal{U}(\lambda, \mu)$ monomial $A$ by

$$\sigma(A) = \rho(A) \left( \lambda \cdot \mu \right) \quad (2.39)$$

where $\lambda$ is a weight associated with $A$ in the following manner: As a monomial of $\mathcal{U}(\lambda, \mu)$, $A$ should be expressible in the form

$$A = a_{1} \times a_{2} \times \ldots \times a_{n} \quad (2.40)$$

where the $a_{j} \in \mathcal{U}(\lambda, \mu)$, $j = 1, \ldots, n$, cannot be further reduced by $\mathcal{U}(\lambda, \mu)$ products. Note that we have omitted to show in the last relation the specific association in the (non-associative) product which makes up $A$; this is immaterial for our purposes. The point is that $n$ factors enter the expansion of $A$ and that number is the weight of $A$.

Now, in (2.40), the factors $a_{j}, j = 1, \ldots, n$, which cannot be further reduced are either of the form

$$a_{k} = \ell_{k} \odot \ell' \quad ; \quad \ell_{k} \in \mathfrak{g} \quad (2.41a)$$

or of the form

$$a_{j} = a_{j} \odot \ell' \quad ; \quad a_{j} \in \mathfrak{g} \quad (2.41b)$$

It can easily be shown, now, that the choice (2.39) is consistent with (2.33). Thus, let $n$ and $m$ be the weights of monomials $A$ and $B$ respectively. Then,

$$\sigma(A \times B) = \rho(A \times B) \left( \lambda \cdot \mu \right)^{n+m-1}$$

$$= \lambda \rho(a) \rho(b) \left( \lambda \cdot \mu \right)^{n+m-2} + \mu \rho(a) \rho(b) \left( \lambda \cdot \mu \right)^{n+m-2} \quad (2.42)$$

On the other hand,

$$\lambda \sigma(A) \sigma(B) + \mu \sigma(B) \sigma(A) =$$

$$= \lambda \rho(a) \rho(b) \left( \lambda \cdot \mu \right)^{n+m} + \mu \rho(b) \rho(a) \left( \lambda \cdot \mu \right)^{n+m} \quad (2.43)$$

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where we have used the result $\rho(A) = \rho(a)$ since $a$ vanishes on $\mathfrak{g}_1$.

Obviously, the right-hand sides of (2.42) and (2.43) are identical and the desired mapping has been constructed.

Several remarks are now in order. To begin with more trivial ones, let us make it clear that all applications we have in mind involve $\mathcal{U}(\lambda,\mu)$ monomials or weight 2. Consequently, we shall gear all specific examples onto such cases. Secondly, we want to stress that our restriction to monomials in the proof of Theorem 2.2 involves no loss of generality since any $\mathcal{U}(\lambda,\mu)$ element can be expressed as a sum of monomials. Consequently, the only complication involved is that one has to deal, separately, with more than one product between monomials.

Of much greater importance is the question concerning the nature of $\mathcal{S}_\lambda$ as an associative algebra of $\mathcal{L}$ monomials. We regard $\mathcal{S}_\lambda$, for all practical purposes, identical to the universal enveloping algebra $\mathcal{A}$ of $\mathcal{L}$. To make this identification transparent, let us first recall that in the case of $\mathcal{S}$ even though its elements are of the form $L = \lambda + \mathbf{R}$, where $\lambda \in T(\mathcal{L})$, one comes to identify $L$ with $\lambda$ and, consequently, one considers $\mathcal{A}$ as an algebra of $\mathcal{S}$ monomials. The aforementioned identification is made formally, via the projection mapping. One may wonder, at this point, as to what the role of $\mathcal{S}$ is (or $\mathcal{A}$ for the case of the UEMA of $\mathcal{L}$) if we are so eager to keep it out of the picture. We shall comment on this issue shortly. To continue with our argument, now, we can also think of $\mathcal{S}_\lambda$ as an (associative) algebra of $\mathcal{L}$ monomials. To see this, consider the simple case of a $\mathcal{U}(\lambda,\mu)$ element $A$ whose form is $a + \mathbf{R}$, $a \in \mathcal{L}$. Then, since $n = 1$, $\sigma(A) = \rho(A) = \rho(a)$

(2.164)

and $\sigma(A) \in \mathcal{S}_\lambda$ can now be associated to the Lie element $a$ through $\sigma$. It is not hard to see how one could continue the argument. Thus, e.g., the term $\rho(\lambda_1)(\lambda_2)$ in (2.36) is an $\mathcal{S}_\lambda$ product of the Lie elements $\lambda_1$ and $\lambda_2$. Finally, for $a \in \mathfrak{g}$ we have $\rho(a) \in a$, i.e., $a$ becomes an $\mathcal{S}_\lambda$ scalar. So one can now write, e.g., $a_1^2 \lambda_1 \lambda_2$ for $\rho(a)(\lambda_1)(\lambda_2)$, $\rho(\lambda_2)(\lambda_2)$, etc.

We have seen, then, that both $\mathcal{A}$ and $\mathcal{S}_\lambda$ can be thought of as algebras of $\mathcal{L}$ monomials. Furthermore, from now on we can follow the usual convention and write, e.g., $ab$ for $(a + \mathbf{R})(b + \mathbf{R})$ (or for $\rho(a)\rho(b)$), $a,b \in T(\mathcal{L})$. We shall not stop here, however, as far as the "identification" process is concerned. We extend the aforementioned conventions to $\mathcal{U}(\lambda,\mu)$ as well. Once again, the mapping makes it possible to identify, e.g., $L_1$ and $L_2$ where $L_1 = L_1 + \mathbf{R}$ and $L_2 = L_2 + \mathbf{R}$, $\mathbf{R} \in \mathcal{L}$. Similarly we can write $L_1 \times L_2$ for $L_1 \times L_2$ and call $L_1 \times L_2$ a non-associative Lie monomial belonging to $\mathcal{U}(\lambda,\mu)$.

Theorem 2.2 finally permits us to write

$$L_1 \times L_2 = \lambda \cdot L_1 \cdot L_2 + \mu \cdot L_1 \cdot L_2$$

(2.45)

where, by $\alpha \times \beta$, we mean $(\alpha + \mathbf{R})(\beta + \mathbf{R})$. From (2.46), the reason for our distinguishing the degree from the weight of a $\mathcal{U}(\lambda,\mu)$ monomial becomes obvious. Thus, scalar factors in a given $\mathcal{U}(\lambda,\mu)$ monomial do contribute a term $(\lambda \cdot \mu)$ in the denominator of (2.38) which is compensated by the factor $(\lambda \cdot \mu)$ entering through (2.39). The monomial $(\lambda_1 + \mathbf{R})(\lambda_2 + \mathbf{R})$ for example, has degree 3 and weight 5.

Next, we want to remark on the difference between the UEMA $\mathcal{U}(\lambda,\mu)$ and the Santilli-Soliani $\mathcal{S}$ mutation algebra $\mathcal{S}(\lambda,\mu)$ (specified in the introduction) because this difference becomes very transparent through Theorem 2.2. Recall that $\mathcal{S}(\lambda,\mu)$ needs, for its definition, the existence of the universal enveloping algebra $\mathcal{A}$; it cannot be constructed directly. Also, recall that $\mathcal{S}(\lambda,\mu)$ and $\mathcal{A}$ are identical as vector spaces but their products are different. On the other hand, if we remember our arguments pertaining to the similarity between $\mathcal{S}$ and $\mathcal{S}_\lambda$, we conclude that $\mathcal{U}(\lambda,\mu)$ is mapped only onto a subspace of $\mathcal{A}$ (a hyperplane in $\mathcal{A}$ parametrized by $\lambda$ and $\mu$). In this distinction, we feel, lies the core of the observation (as will be developed in the next section) that a "total" mathematical framework in which $\mathcal{U}(\lambda,\mu)$ belongs should be non-Lie. On the contrary, $\mathcal{U}(\lambda,\mu)$ is identical to $\mathcal{A}$ as a vector space and, as such, it would be difficult to see that it would lead to a radically non-Lie framework (e.g., one for which the underlying group manifold is non-Lie).

Our final comment regards the exact role which $\mathcal{S}(\mathfrak{g}')$ plays in the enveloping scheme. It seems natural to ask why one cannot work with $T(\mathfrak{g}) (T(\mathfrak{g}))$ exclusively and, thus, encounter Lie elements directly? The fact is that without $\mathcal{S}(\mathfrak{g}')$ no reordering would have been meaningful. More specifically, a term such as $e_1 \otimes e_2$ would be completely different from $e_2 \otimes e_1$; these two expressions could not have been related in any manner. The P-B-W theorem...
constitutes the culmination of the role of $\mathfrak{g}(\mathfrak{g})$ in the formation of the enveloping algebra (UEA) of a Lie algebra.

We shall use the result of theorem 2.2, in particular through its more straightforward form (2.15), in both the next section and the appendix.

III. GENERAL PHYSICAL CONSIDERATIONS AND AN INTERESTING SPECIFIC EXAMPLE

The contact between Lie groups (algebras), on one hand, and field theory, on the other, comes through representation theory. Basically, one wants to relate observed (or conjectured) symmetries to the structure of a Lie group. One then works with field variables that belong to a space on which the assumed symmetry (Lie) group acts through a representation. As is well known, the elements of the Lie algebra of a Lie group (4) correspond (via their representation) to the generators of infinitesimal transformations. Accordingly, algebra representations are not interesting enough by themselves. However, a relation between an element of the Lie algebra and an element of the Lie group is established via the so-called exponential mapping. The latter can be specified in an abstract topological manner, but its more practical and readily applicable form is obtained by referring to the universal enveloping algebra of $\mathfrak{g}$. Secondly, the existence of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is of vital practical importance when it comes to representations of $\mathfrak{g}$. To appreciate the last statement consider what is usually done when one deals with a representation $\rho$ of a Lie algebra $\mathfrak{g}$ on some vector space. One automatically sets

$$\rho([\mathfrak{g}, \mathfrak{g}]) = \rho(\mathfrak{g}) \rho(\mathfrak{g}) - \rho(\mathfrak{g}) \rho(\mathfrak{g}) .$$

(3.1a)

Similarly, when one considers two successive infinitesimal transformations one assumes that, e.g., $\rho(\mathfrak{g}) \rho(\mathfrak{g})$ makes sense as representing an $\mathfrak{g}_2$-transformation followed by an $\mathfrak{g}_2$-transformation. In both the above examples, it is the framework of the universal enveloping algebra that needs to be employed; otherwise, the object $\mathfrak{g}_2 \mathfrak{g}_3$ would not make any sense. Fortunately, the homomorphism which always exists between $\mathfrak{g}$ and $\mathfrak{g}_3$ ensures that (3.1) can, indeed, be used.

To be more precise, let us recall certain important aspects from the theory of representations of a universal enveloping algebra. In particular, we know that the set of all representations of $\mathfrak{g}$ on a vector space $V$ is in one-to-one correspondence with the set of all representations of $\mathfrak{g}$ on $V$. $\mathfrak{g}$ denotes the universal enveloping algebra of $\mathfrak{g}$). The proof hinges on the following observation. If $\rho$ is a representation of $\mathfrak{g}$, we define a representation $\tilde{\rho}$ on $\mathfrak{g}(\mathfrak{g})$ which agrees with $\rho$ on $\mathfrak{g}$, i.e. $\tilde{\rho}(\mathfrak{g}) = \rho(\mathfrak{g})$ for $\mathfrak{g} \in \mathfrak{g}$. This representation is unique if we set, e.g. $\tilde{\rho}(\mathfrak{g}) \tilde{\rho}(\mathfrak{g}) = \rho(\mathfrak{g}) \rho(\mathfrak{g})$, etc. Thinking, now, of a Lie algebra as the tangent space at the identity element of the group manifold, the Lie product $[\mathfrak{g}_1, \mathfrak{g}_2]$ can be identified with $(\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5)$, where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are vector fields over the manifold and $\mathfrak{g}_3$ denotes the tangent vector at $e$ corresponding to the vector field $\mathfrak{g}_3$. From these remarks it follows that:

$$\tilde{\rho}(\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5) \cdot \rho(\mathfrak{g}_1) \rho(\mathfrak{g}_2) \rho(\mathfrak{g}_3) \rho(\mathfrak{g}_4) \rho(\mathfrak{g}_5) = 0.$$

(3.1b)

In other words, $\tilde{\rho}$ vanishes on the ideal $\mathfrak{g}$. Finally, the representation $\rho$ is extended to a representation $\tilde{\rho}$ of $\mathfrak{g}$ by $\tilde{\rho} = \tilde{\rho}$, where $\pi$ is the projection mapping of $\mathfrak{g}(\mathfrak{g})$ onto $\mathfrak{g}(\mathfrak{g})/\mathfrak{g}$). Similar arguments hold for the converse route.

Now, $\mathfrak{g}$ is an associative algebra. As such, it has a faithful representation by linear transformations on a certain vector space. It follows that any Lie algebra also has a faithful representation by linear transformations. This is a result which is often used in connection with local field theory.

Having stressed the close connection between the representations of a Lie algebra and those of its enveloping algebra we want to comment now on the connection between the enveloping algebra and the Lie group. As is well known, the exponential mapping relates an element $X$ of the Lie algebra $\mathfrak{g}$ to the element $\exp X$ of $\mathfrak{g}$. In its abstract definition the mapping $\exp: \mathfrak{g} \rightarrow \mathfrak{g}$ sends lines in $\mathfrak{g}$ (i.e. vectors along a given direction) to geodesic curves on the group manifold. However, a more practical approach, once again, is arrived at if we refer to the enveloping algebra and think of $\exp X$ in the conventional way, i.e.

$$\exp X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots$$

(3.2)

Now, (3.2) really makes sense if $\exp X$ stands for a representation of the corresponding group element and $1, X, X^2$, etc. are representations of corresponding enveloping algebra elements. Then, we can interpret (3.2) as follows: $\exp X$ is a (finite) group transformation which is the sum of a series of infinitesimal transformations generated by $X$. 

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The general conclusion from the above discussion is that the universal enveloping algebra \( U(\mathcal{L}) \) of a Lie algebra is of indispensable practical value when it comes to Lie representations. Furthermore, the framework of \( U(\mathcal{L}) \) is necessary for expanding the action of a group element in terms of a series of infinitesimal transformations. In the case of a UEMA of a Lie algebra, one does not know whether the parallel to a group, standing in relation to \( U(\mathcal{L}) \) as \( G \) stands in relation to \( \mathcal{L} \), exists and, if it does, what its mathematical structure is. Clearly, if one wants to see a \( U(\mathcal{L}) \) mathematical scheme finding applications in physics one would have to have the parallel of the Lie group. Since, now, we have built \( U(\mathcal{L}) \) from the Lie algebra \( \mathcal{L} \), one may be tempted to think that perhaps the group structure of \( G \) can support \( U(\mathcal{L}) \) as its non-associative, \((\lambda,\mu)\)-parametrized enveloping algebra. That this is not so, can be seen very easily if we consider the Lie group product
\[
\exp tX \cdot \exp tY = \exp \left( t(X + Y) + \frac{1}{2} [X,Y] + O(t^3) \right),
\]
where \( O(t^3) \) contains third-order terms in \( t \). For convenience let us assume \( |t| \ll 1 \) so that \( O(t^3) \) will contain negligible terms. In (3.4) \( [X,Y] \) stands for the enveloping algebra expression \( XY - YX \). In physics one would have to have the parallel of the Lie group. Since, now, we have built \( U(\mathcal{L}) \) from the Lie algebra \( \mathcal{L} \), one may be tempted to think that perhaps the group structure of \( G \) can support \( U(\mathcal{L}) \) as its non-associative, \((\lambda,\mu)\)-parametrized enveloping algebra. That this is not so, can be seen very easily if we consider the Lie group product
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Arguing in a cruder way, we can say that the UEMA of a Lie algebra \( \mathcal{L} \), even though constructed from the latter, does not go naturally with \( \mathcal{L} \) or with the Lie group \( G \). It seems reasonable that one should try to find (non-Lie) structures \( \mathcal{L}' \) and \( \mathcal{G}' \) which should be related to \( \mathcal{L} \) and \( G \), respectively, in some simple way - much as \( U(\mathcal{L},\mu) \) is related to \( \mathcal{L} \). Only then would one have a completely non-Lie framework which, in principle, can be employed in place of the customary Lie framework. In such a case, the parameters \( \lambda \) and \( \mu \) would have to be given some physical interpretation. We shall make some relevant speculations on this matter in the next section.

We shall not, in this paper, attempt to discover the parallel of \( G \) for a UEMA. Instead, we shall turn our attention to representations of the UEMA itself. We shall think of them as infinitesimal (non-Lie) transformations. Since a UEMA is a non-associative algebra, we do not expect that it has linear representations on a vector space. That is, if we substitute the UEMA framework for the universal enveloping algebra without changing the underlying group, we face difficulties. In particular, the Lie bracket \([X,Y]\) in (3.4) will be substituted by \( [X,Y]' = (X-\lambda)[X,Y] \). However, such a redefinition shifts the product \( \exp \left( t(X + Y) + \frac{1}{2} [X,Y] + O(t^3) \right) \) (3.3)

where \( O(t^3) \) contains third-order terms in \( t \). For convenience let us assume \( |t| \ll 1 \) so that \( O(t^3) \) will contain negligible terms. In (3.4) \( [X,Y] \) stands for the enveloping algebra expression \( XY - YX \). If, now, we substitute the UEMA framework for the universal enveloping algebra without changing the underlying group, we face difficulties. In particular, the Lie bracket \([X,Y]\) in (3.4) will be substituted by \( [X,Y]' = (\lambda,\mu)[X,Y] \). However, such a redefinition shifts the product \( \exp tX \cdot \exp tY \) (see Fig.1). Indeed, \( t(X + Y) + \frac{1}{2} [X,Y] \) will not lie along the direction of \( t(X + Y) + \frac{1}{2} [X,Y] \) in the vector space \( \mathcal{L} \). Accordingly, the exponential curve of the first expression will not coincide with that of the second on the group manifold. Hence, to say the least, the adoption of the UEMA in the place of the universal enveloping algebra of \( \mathcal{L} \) alters the group structure of the underlying manifold.

We devote the rest of this section to an application of interest and, for this purpose, we restrict ourselves to the Lie group \( U(1) \). We recall that local \( U(1) \) transformations, within the conventional Lie framework, lead to the introduction of an electromagnetic field potential \( A_\mu \) in a very natural way. To be specific, suppose we have two charged scalar fields, \( \phi \) and \( \phi^* \), in a theory which is invariant under local \( U(1) \) transformations. The electromagnetic field potential \( A_\mu \) is then introduced in order to covariantize the derivative \( \partial_\mu \). The Lagrangian density
\[
\mathcal{L} = e^{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \phi^* \partial^2 \phi - V(\phi^*)}
\]
where
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]
and
\[
V(\phi^*) \text{ is a polynomial expression in } \phi^*, \text{ is invariant under gauge transformations of the second kind}
\]
\[
\phi \rightarrow e^{i\alpha(x)} \phi \quad ; \quad \phi^* \rightarrow e^{-i\alpha(x)} \phi^*
\]

We shall not, in this paper, attempt to discover the parallel of \( G \) for a UEMA. Instead, we shall turn our attention to representations of the UEMA itself. We shall think of them as infinitesimal (non-Lie) transformations. Since a UEMA is a non-associative algebra, we do not expect that it has linear representations on a vector space. That is, if we substitute the UEMA framework for the universal enveloping algebra without changing the underlying group, we face difficulties. In particular, the Lie bracket \([X,Y]\) in (3.4) will be substituted by \( [X,Y]' = (\lambda,\mu)[X,Y] \). However, such a redefinition shifts the product \( \exp tX \cdot \exp tY \) (see Fig.1). Indeed, \( t(X + Y) + \frac{1}{2} [X,Y] \) will not lie along the direction of \( t(X + Y) + \frac{1}{2} [X,Y] \) in the vector space \( \mathcal{L} \). Accordingly, the exponential curve of the first expression will not coincide with that of the second on the group manifold. Hence, to say the least, the adoption of the UEMA in the place of the universal enveloping algebra of \( \mathcal{L} \) alters the group structure of the underlying manifold.
In particular, (3.6a) implies that \( \phi^* \phi = \phi^\dagger \phi \).

Suppose, now, that we choose to work within the non-Lie framework of a UEMA. (We re-emphasize that, since we lack the full mathematical machinery, we confine ourselves to infinitesimal transformations.) A new possibility now arises. We start with the Lagrangian density

\[
\mathcal{L} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} \lambda^2 \phi^* \phi ,
\]

i.e. we are specializing to the simple case \( V(\phi^* \phi) = \frac{1}{2} \lambda^2 \phi^* \phi \). If we want our theory to be invariant under \( U(1) \)-related local transformations and we work within the framework of the UEMA of \( U(1) \), we could consider the following infinitesimal transformations:

\[
\phi \rightarrow \phi + i \lambda(x) \phi \quad \text{and} \quad \phi^* \rightarrow \phi^* - i \beta(x) \phi^* .
\]

We shall comment later on the phase difference in the way \( \phi \) and \( \phi^* \) transform.

Now, \( \alpha(x) \) and \( \beta(x) \) are actually multiplied by the generator of infinitesimal \( U(1) \) transformations.\(^{19}\) As is the usual practice, however, the aforementioned generator has not been denoted explicitly, since it can be scaled via \( \alpha(x) \) (and \( \beta(x) \)) to the identity operator (due to the 1-dimensionality of \( U(1) \)). But, whereas the identity operator of the usual (Lie-framework) representation has the trivial property

\[
(\rho(1) \rho(1)) = \rho(1) \rho(1) = \rho(1) ,
\]

within the UEMA framework we have

\[
\rho(1) \times \rho(1) = \rho(1 \times 1) = \rho((1 + i)1) = (\lambda + \mu) \rho(1) .
\]

We see, then, that (neglecting higher order terms in the infinitesimals)

one now obtains

\[
\phi^* \rightarrow (\lambda + \mu) \phi^* + i(\lambda + \mu)(\alpha - \beta) \phi^* = (\lambda + \mu)[1 + i(\alpha(x) - \beta(x))] \phi^* .
\]

Consider, first, the case \( \alpha = \beta \). One sees that the term \( i(\alpha(x) - \beta(x)) \phi^* \) in (3.7) must be replaced by \( \frac{1}{2} (\alpha(x) - \beta(x)) \phi^* \). At the same time, one has to make the replacement \( \partial_{\mu} \phi \rightarrow \partial_{\mu} \phi^* \), where

\[
\partial_{\mu} \phi^* = \frac{1}{i\mu \lambda} (\partial_{\mu} - i e A_{\mu}) \phi^* \quad \text{and} \quad \partial_{\mu} \phi = \frac{1}{i\mu \lambda} (\partial_{\mu} + i e A_{\mu}) \phi .
\]

Thus, one obtains the regular formalism of electrodynamics with a redefinition of the matter (scalar) fields.

Suppose, on the other hand \( \alpha = \lambda \neq 0 \). One could still obtain an invariant \( \phi^* \phi \) term by choosing

\[
(\lambda + u)(1 + iy(x)) = 1 ,
\]

or, for infinitesimal \( \gamma \),

\[
\lambda + u = e^{-i\gamma} ,
\]

It is straightforward to see that the same factor, i.e. \( (\lambda + \mu)e^{-i\gamma} \), will appear in front of \( \partial_{\mu} \phi^* \). By (3.14) this factor will, once again, disappear. The big difference is that we now need two gauge fields. In particular, we set

\[
\partial_{\mu} \phi = [\partial_{\mu} - i e A_{\mu}] \phi ,
\]

where \( A_{\mu} \) transforms according to

\[
A_{\mu} = A_{\mu} + \frac{1}{\mu} \partial_{\mu} \alpha(x) ,
\]

and

\[
\partial_{\mu} \phi = [\partial_{\mu} + i e B_{\mu}] \phi ,
\]

where \( B_{\mu} \) transforms according to

\[
B_{\mu} = B_{\mu} + \frac{1}{\mu} \partial_{\mu} \beta(x) .
\]

Even though the above scheme is quite unorthodox, it bears striking similarity to the (phenomenological) one recently proposed by Hsu.\(^{6}\) In his version of superweak-interaction-interpretation of CP violation by the \( K^0 \) system, Hsu's phenomenological Lagrangian is more complicated in that it requires the presence of spinor fields as well. Let us isolate, for the sake of comparison, the vector-scalar part of Hsu's Lagrangian which reads:

\[
\mathcal{L}_{HSU} = -\left( \partial_{\mu} + i \xi [A_{\mu} + i \xi C_{\mu} K^0] K^0 \right) K^0 F_{\mu \nu} F^{\mu \nu} - \left( \partial_{\mu} + i \xi [A_{\mu} + i \xi C_{\mu} K^0] K^0 \right) K^0 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{4} C_{\mu \nu} C^{\mu \nu} + (\text{terms pertaining to choice of gauge}),
\]

\[
(3.19)
\]

\[25\]
where \( F \) is a complex and \( \beta \) a real number and the meanings of \( F_{\nu\mu} \) and \( C_{\mu
u} \) are evident. Furthermore, \( A_j \) and \( C_j \) transform according to

\[
A_j \rightarrow A_j + F_{\nu\mu} \frac{\partial\Lambda(x)}{\partial F_{\nu\mu}}; \quad C_j \rightarrow C_j - F_{\mu\nu} \frac{\partial\Lambda(x)}{\partial F_{\mu\nu}} \quad (3.20)
\]

and the field \( K^0 \) according to

\[
K^0 \rightarrow e^{iA(x)} K^0 \quad (3.21)
\]

By rearranging our \( A_j \) and \( H_j \) one should obtain Hsu's Lagrangian with the redefined vector fields obeying (3.20). Note that (3.20) implies that our \( \gamma \) is constant, i.e. that \( a(x) \) and \( \beta(x) \) are functions differing by a constant.

As mentioned before, Hsu's model requires the presence of spin-3 fields also. No added difficulty really enters by introducing such fields. Indeed, if we were to require that \( \psi(x) \) and \( \overline{\psi}(x) \) have the transformation properties

\[
\psi \rightarrow e^{ia(x)} \psi; \quad \overline{\psi} \rightarrow e^{-ib(x)} \overline{\psi} \quad (3.22)
\]

then the choice (3.14) should render the mass term \( \mu \overline{\psi}\psi \) invariant and should also effect the replacement \( \psi \rightarrow \psi' = e^{iA(x)} \psi \), where

\[
\psi' = (\beta_j + i\alpha_j) \psi \quad (3.23)
\]

and

\[
\overline{\psi'} = (\beta_j - i\alpha_j) \overline{\psi} \quad (3.24)
\]

The same redefinition of \( H_j \) and \( H_j \) suggested following (3.21) simultaneously brings the spinor terms to the form given by Hsu. All one needs to include are the terms pertaining to the choice of gauge and the full Lagrangian density proposed for CP violation by the \( K^0 \) system should emerge.

The drawback from all this is that we have restricted ourselves to infinitesimal transformations since we lack the existence of the analogue to a group structure. Still, it should become evident from this example that there could, indeed, be merits in the whole UEMA scheme.

We close our present discussion with some remarks on the phase difference \( \gamma(=\alpha - \beta) \) between the way the particle transformation and that of its charge conjugate (i.e., relations (3.8a,b)). Presumably such a phase difference should be connected with a "charge" violation which Hsu already suggests in Ref. 6. To follow Hsu's reasoning, his theory suggests that the charge of all particles fulfills the relation

\[
\text{"charge"} = q + \lambda S \quad (3.25)
\]

with \( \lambda \) a very small (constant) parameter, \( S \) the strangeness and \( q \) standing for \( \pm e \) or 0. Furthermore, the "charge" in (3.25) is conserved implying also conservation of strangeness.

Within our theoretical scheme, \( \lambda \) of the last formula should be directly related (if not identical) to our \( \gamma \). Most probably the relation between Hsu's additional charge and our charges \( \alpha, \beta \) should be something like

\[
\frac{\alpha - \beta}{\lambda} = \lambda S \quad (3.26)
\]

It is, needless to emphasize, premature to discuss the details of Hsu's scheme as, e.g., the various numbers entering and/or proposed. We should mention, however, his call for experimental tests which should look for anomalous phenomena associated with \( K^0 \)-decay in the electric field. Clearly, if such experiments lend support to Hsu's model, our theoretical scheme should serve as one of the possible alternatives into which this particular broken symmetry scheme fits naturally.

IV. DISCUSSION AND SPECULATIONS

As we have already mentioned in the introduction, many speculations can be made concerning the use of two parameters one has in hand. We note that, for our case, the two parameters are presumably connected with a general non-Lie symmetry scheme which, hopefully, can be employed for physical descriptions. In the previous section, we have indicated that the corresponding (non-Lie) framework is far from being complete. On the other hand, we have also shown, in Sec. III, that one unorthodox scheme which can be deduced from UEMA considerations bears striking resemblance to a phenomenological theory proposed for the explanation of a very important problem in elementary particle physics. Accordingly, we are encouraged to proceed and comment briefly on a few possibilities of UEMA relevance to physics even though we are aware of the fact that a complete UEMA formalism is not as yet fully developed.

Our most immediate thoughts turn to current algebra. Indeed, some recent approaches aimed towards the better understanding of symmetry breaking may have a natural position within the realms of the UEMA of SU(3). We are
referring to the deformed algebra approach of Oehme \(^\text{(21)}\) and Mathur \(^\text{(22)}\) as well as the closely related "weighted" SU(3) transformations of Rest and Welling \(^\text{(23)}\). These attempts lead to insights concerning the Cabibbo angle as well as several good predictions connected with \(X_{u_d}\) form factors. We shall indicate briefly that there exist possibilities of incorporating such approaches into the UEMA scheme. In passing, we would like to remark that we hope the relevance of the UEMA machinery will be, once fully developed, precisely its capacity to incorporate apparent deviations from an ideally perfect symmetry within a background where the aforementioned symmetry firmly resides. In this sense, the deformed algebra ideas, for example, constitute exactly the kind of approach that fits into the ideas behind what we hope to be the physical relevance of the UEMA scheme. Perhaps the most commonplace term "hidden symmetry", which is also becoming of use, best reflects the content of the last two statements. (Indeed, it may be appropriate to attribute the ideal symmetry, which always resides in the background, to some hidden (particle?) structure of unobservable fields and reserve the somewhat looser symmetry framework (e.g. UEMA) to reflect what is observed.)

So as not to make our discussion too involved we shall concentrate, for the most part, on Oehme's scheme \(^\text{(21)}\) which is the simplest of the three we have mentioned above. Consider a chiral octet of SU(3) currents \(J_{\mu} = V_{\mu} + A_{\mu}\) whose charges obey chiral SU(3) + SU(3) commutation rules at equal times (ET). Suppose we believe the chiral SU(2) x SU(2) part of the symmetry to be exact (\(m_\pi = 0\) and absence of electromagnetic effects). According to Oehme, a very simple way to implement this state of affairs is to redefine the currents by

\[
\begin{align*}
J_{\mu}^\alpha &= \sum_{a=1}^{3} (k_a)_{\mu}^\alpha \\
J_{\mu}^7 &= \sum_{a=4}^{6} (k_a)_{\mu}^7 \\
J_{\mu}^8 &= \sum_{a=7}^{8} (k_a)_{\mu}^8
\end{align*}
\]

Accordingly, equal-time charge commutators remain the same except for those among charges with indices running between \(4\) and \(7\). In particular,

\[
[Q_{\mu}^a, Q_{\nu}^b] = \epsilon_{\mu \nu \lambda} P_{\lambda}^a, \quad a = 1,2,3,8
\]

i.e. a factor of \(\epsilon_{\nu \lambda}^\mu\) enters on the right-hand side. This procedure makes it possible to go ahead with a computation of the Cabibbo angle and, more important, to obtain some insight as to its meaning.

Now, \((4.2)\) induces some thoughts in connection with the UEMA formalism of SU(3). Suppose, in general, that we deviate from a Lie to the corresponding UEMA framework in a minimal manner, by which we mean that we use \(U_L(\lambda,\mu)\) to replace \(Q_{\mu}^a\). But whereas \(Q_{\mu}^a\) can be put into natural correspondence (homomorphism) with the Lie algebra \(\mathcal{L}\), \(U_L(\lambda,\mu)\) corresponds in the same manner to the Lie algebra \((\lambda,\mu)\mathcal{L}\). By \((\lambda,\mu)\mathcal{L}\) we mean the Lie algebra \(\mathcal{L}'\) with elements formally identical to those of \(\mathcal{L}\) but with Lie product

\[
[Q_{\mu}^a, Q_{\nu}^b] = \epsilon_{\nu \lambda}^\mu (Q_{\lambda}^a Q_{\lambda}^b)
\]

It is not immediately obvious from \((4.3)\) whether the parameter \(\lambda,\mu\) is of any special significance. On one hand, \((4.3)\) looks like a redefinition of the basic elements of \(\mathcal{L}\) or, alternatively, of its structure constants. Thus, the SU(3) equal-time commutator

\[
[Q_{a}^\mu, Q_{b}^\nu] = \epsilon_{\mu \nu \lambda} C_{\lambda}^a
\]

with \(a, b, c\) running through all SU(3) indices. In turn, \((4.5)\) is equivalent to

\[
[Q_{a}^\mu, Q_{b}^\nu] = \epsilon_{\mu \nu \lambda} C_{\lambda}^a
\]

On the other hand, the study of subalgebras - or, more generally, of inner structures - of UEMA may open up possibilities of the kind described by \((4.2)\). For example, the subalgebra SU(2) of SU(3) has its corresponding \((\lambda,\mu)\) SU(2) image. Suppose, then, we build a \((\lambda,\mu)\) generalization of SU(3) in two stages, i.e. we first construct \((\lambda,\mu)\) SU(2) and then, through a second generalization \((\lambda,\mu)\), we introduce the full \((\lambda,\mu)\) generalised SU(3). In that case, we would have two parameters, \(a = \lambda, \mu\) and \(\mu = \lambda, \mu\), which could be used to obtain commutation relations such as those given by \((4.2)\). In fact, if it is our contention (and/or ambition) that the \((\lambda,\mu)\) parametrisation measures deviations from exact symmetries, we would (physically) expect a different \((\lambda,\mu)\) content for the SU(2) x SU(2) part of the algebra of currents than for the whole of SU(3) x SU(3) - as observation warrants. Suppose, then, that \((4.5)\) is used to describe the full SU(3) part of the algebra of currents, whereas for the isospin restriction we accept a generalisation \((\lambda,\mu)\) closer to \((1,0)\) and reflected by the relations

\[
[Q_{\mu}^a, Q_{\nu}^b] = \epsilon_{\mu \nu \lambda} C_{\lambda}^a
\]
In that case, \(Q_8^a, Q_2^a\) and \(Q_3^a\) would not be the true isospin charges \((R_3)\).

Rather, there should be a relation between the two sets given by

\[Q_k = \epsilon^{R_k}, \quad k = 1, 2, 3, \quad (4.8)\]

where \( \beta = \alpha/\alpha' \).

Similarly, if the \((\lambda, \mu)\) generalization of the U(1) symmetry associated with the hypercharge was effected through \((\lambda, \mu)\) - presumably closer to \((1,0)\) than \((\lambda, \mu)\) - then \(Q_8\) would be related to the hypercharge charge \(Y\) by

\[Q_8 = \gamma Y, \quad (4.9)\]

where \( \gamma = \alpha/\alpha' \).

In particular, Gehme's choice is \( \alpha = a = 1 \) and \( \beta = \gamma = a \); Mathur's work \(26\), on the other hand, deals with \( 1 \neq a \neq \alpha \).

We also wish to say a few words on the "weighted" SU(3) transformation approach of Rest and Welling, \(23\) which is very closely related to the deformed algebra ideas. The difference is that these two authors consider an octet of fields transforming under SU(3). Accordingly, one now has commutation relations between SU(3) generators \(T_a\) and field operators \(\varphi_a\), \(a = 1, \ldots, 8\). Once more, if the observed octet of fields does not transform under the exact SU(3) rules \(\left[ T_a, \phi_b \right] = i \epsilon_{abc} \phi_c\), then one can start thinking about a more complicated SU(3) or SU(2) behaviour of the \(\phi_a\)'s. Since the hope still remains that the exact SU(3) is hidden and is consequently obeyed by unobserved constituent fields \(\varphi_a\), Rest and Welling are thereby led to construct the \(Q_a\) from the \(\varphi_a\) in such a manner that the former transform (in a certain limit) by

\[\[ T_a \phi_b = i \epsilon_{abc} \phi_c. \quad (4.10)\]

The \(Z_k\), \(k = 1, \ldots, 8\), are parameters which are taken to be \(Z_1 = Z_2 = Z_3 = 7\), \(Z_4 = 2Z_2 = 2Z_3 = 2a = 2 + V = \frac{Z}{2} + \frac{Z}{2} = \frac{Z}{2} + \frac{Z}{2} = 1\). One readily finds that the \(\varphi_a\) in (4.10) transform under exact SU(3) conditions unless the SU(3) generator \(T_a\) has an index between \(1 \leq a \leq 7\). In other words, the factor \(\varphi_a\) in (4.10) is different from unity only when \(k \leq a \leq 7\). Thus, the Rest and Welling scheme is similar (in the limit where (4.10) is valid) to that of Gehme. More exactly, it corresponds to the UEMA description \(a = a \neq 1\). (We have used the notation already introduced when we examined the possible UEMA content of Gehme's approach.) It may be worth mentioning that the scheme of Rest and Welling leads to some nice predictions with regard to \(K_L\) form factors. \(24\)

We feel encouraged by the fact that the matters we have discussed in the last few paragraphs have a natural position within the UEMA framework. On the other hand, all our arguments concerning connections with the UEMA scheme must be considered formal at this point. Further work is clearly needed with regard to inner structures (subalgebras and so on) of the UEMA of a Lie algebra.

Finally, the presence of the two parameters induces some more far reaching thoughts which are motivated by recent theoretical developments in elementary particle physics. We are referring to the ideas of Lee and Wick \(25\) on one hand, and the work of Kirzhnits and Linde \(26\) and of Weinberg \(27\) on the other. What emerges from these studies is the mutability of physical laws, or, perhaps more appropriately, the existence of domains in the universe which may exhibit different characteristics of symmetry behaviour. Both of the aforementioned trends of thought are intimately connected with the question of what is the true meaning behind spontaneous symmetry-breaking mechanisms.

Of course, it is too early to profess deep understanding about such matters but we would like to ponder for a minute on a speculative significance of a scheme such as the UEMA in connection with such ideas. Thus, let us take note of the fact that intrinsic symmetries seem to form a hierarchy (e.g., U(1), SU(2), SU(3)) whereby the more complex the intrinsic symmetry the more it appears to be broken. As we have already argued in this section, the UEMA parametrization \((a, \mu)\) can conceivably measure deviations from a perfect (Lie) symmetry through its departure from \((1,0)\). Now, if \(a\) and \(\mu\) are local functions of space-time (presumably very slowly varying), then one can think of a space-time "evolution" of intrinsic symmetries - an idea which comes close to the thinking of Kirzhnits and Linde as well as Weinberg. Perhaps, also, one could find a relation between, say, \(\mu = 0\) and the non-vanishing expectation value \(\langle \psi(x) \rangle\) in an abnormal domain of Lee and Wick.

It would carry us too far to continue such speculations. In fact, it would be presumptuous to go on doing so when a complete non-Lie framework for symmetry descriptions is not as yet available. It would be interesting, in any case, to pursue unorthodox schemes such as the one we have introduced here. For the particular case of a UEMA the important next step is to look for the corresponding "mutation" group.

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In this appendix we shall be concerned with an application of the P-B-W theorem for a UEMA, proved in Sec. II, pertaining to monomials of degree 2. For this particular case the \( W(\lambda, \mu) \) expansion does not contain any \( \Omega \) terms.

The most practical implication of the P-B-W theorem, in connection with universal enveloping algebras is the following. Let \( \{e_1, \ldots, e_j\} \) be an ordered basis for \( \mathcal{L} \). Then, the element 1 together with all standard monomials \( e_1^j, e_2^j, \ldots, e_k^j \), \( j_1 \leq j_2 \leq \cdots \leq j_k \), form a basis for the universal enveloping algebra of \( \mathcal{L} \). This remark leads, naturally, to a Gell-Man-Okubo mass formula for, say, an SU(3) octet (i.e. \( \mathcal{L} = \text{SU}(3) \)). Indeed, the well-known formula

\[
\mathcal{T}_\nu^\mu = a_\nu^\mu + b A_\nu^\mu + c \sum_j A_{\mu}^j A_{\nu}^j,
\]

(A.1)

where \( T_\nu^\mu \) is an SU(3) tensor operator (of 2nd rank), is a natural consequence of the fact that a general monomial of degree 2 can be expanded as a linear combination of standard monomials of degree 2 and less. Thinking, in particular, in terms of a matrix representation of \( \mathcal{L} \), it follows that

\[
A B = a I + \sum b_{ij} E_j^i + c \sum_{ij} E_i^j E_j^i,
\]

(A.2)

where \( A, B \) as well as the \( E_j \) are matrix representations of Lie elements (the \( E_j \), in particular, correspond to matrix representations of the basis of \( \mathcal{L} \)). Not thinking in terms of the universal enveloping algebra \( \mathcal{L} \), \( A B \) corresponds to a second-rank (SU(3)) tensor. Accordingly, (A.1) is a tensorial version of (A.2).

We want to derive the analogue of the Gell-Man-Okubo mass formula within the framework of the UEMA of SU(3). To this effect we shall neglect mixing effects between multiplets (even though such a mixing is very pronounced in some cases, e.g., \( \Phi \)-\( u \) mixing). Let us, following Okubo, confine our considerations to an SU(3) octet (baryon octet for simplicity) and assume that the mass operator transforms as an SU(3) scalar plus a \( Y + T = 0 \) of an SU(3) octet. In making this assumption we have excluded the possibility of mixing between particles belonging to different representations of SU(3) as already remarked.

Choosing appropriate SU(3) generators \( A_i^j \), \( i,j = 1,2,3 \), with \( \frac{A}{2} A_i^j A_i^j = 0 \) corresponding to the \( E_j \) of Eq. (A.2) it follows that \( \mathcal{T}_3^3 \) (\( Y = 0 \)) has the form

\[
\mathcal{T}_3^3 = a e_3^3 + b_3 \left( A_i^3 A_i^3 - \frac{1}{3} c_3^3 \right),
\]

(A.3)

where \( c_3 \) is the second-order Casimir operator for SU(3). One can obtain the Gell-Man-Okubo mass formula from (A.3) in a straightforward manner.

Returning now to our UEMA, the expansion (A.2) is replaced by an expansion in terms of standard monomials with respect to \( \mathcal{F} \). Let us choose for SU(3) the same basis as that leading to (A.3). We must first make some adjustments. The standard monomial \( A_i^j A_i^j \) will be replaced by

\[
A_i^j A_i^j \rightarrow A_i^3 A_i^3 = \lambda A_i^3 A_i^3 + \mu A_i^3 A_i^3.
\]

(A.4)

Using the commutation relation for the \( A_i^j \),

\[
[A_i^j, A_k^l] = \delta_j^l A_k^i - \delta_i^l A_k^j,
\]

(A.5)

we obtain

\[
A_i^3 A_i^3 = (\lambda + \mu) A_i^3 A_i^3 + \mu (A_i^3 A_i^3) - 2 \mu A_i^3.
\]

(A.6)

Substituting \( A_i^3 + A_i^3 = -A_i^3 \), we finally have

\[
A_i^3 A_i^3 = (\lambda + \mu) A_i^3 A_i^3 - 3 \mu A_i^3.
\]

(A.7)

Next we consider the Casimir operator \( C_2 \) which is given by

\[
C_2 = \frac{1}{2} A_i^3 A_i^3.
\]

(A.8)

The replacement of (A.8) for the case of the UEMA is

\[
C_2 \rightarrow C_2 = \frac{1}{2} (\lambda A_i^3 A_i^3 + \mu A_i^3 A_i^3) = (\lambda + \mu) C_2.
\]

(A.9)
Obviously, since $\lambda$ and $\mu$ belong to the field of scalars $\mathbb{F}$ over which the algebra of $SU(3)$ is defined as a vector space, $(\lambda + \mu)C_2$ is an $SU(3)$ scalar inasmuch as $C_2$ is.

Recalling that

\[ A_j^2 = -Y; \quad A_j^1 A_j^1 = C_2 - \frac{1}{3} Y^2 + \frac{3}{2} Y, \]

where $\frac{1}{2}$ is the isospin and $Y$ the hypercharge operator, we finally conclude that

\[ T_3 \cdot -\alpha Y + b \left[ (\lambda + \mu) \left\{ C_2 - \frac{1}{3} Y^2 + \frac{3}{2} Y \right\} + \frac{\alpha}{3} Y - (\lambda + \mu) C_2 \right], \]

which gives

\[ \Delta M = b \frac{2(\lambda + \mu)}{3} C_2 + \left[ \frac{3}{2} b (\lambda + \mu) - \alpha \right] Y + (\lambda + \mu) \left[ \frac{\alpha}{3} Y - T(T+1) \right]. \]

Note that we recover the Gell-Mann-Okubo mass formula when $\lambda = 1, \mu = 0$.

A similar formula, not identical to the above, however, has been given in within the framework of the mutation algebra of an enveloping algebra, i.e. $\mathcal{A}(\lambda, \mu)$.

REFERENCES AND FOOTNOTES


2) The usual way of regarding $[a, b]$, i.e. $[a, b] = ab - ba$, presupposes the existence of the product $ab$. Here, we want to remain quite general in that the product $[a, b]$ is considered to have been introduced \textit{a priori} by itself.


4) Briefly, an algebra $A$ is Lie admissible if its product can be utilized, via a commutator, to introduce a new product satisfying the Lie properties (see Sec. II for an explicit formulation). Throughout this paper we adopt the Lie admissibility in Santilli's sense. It is specified as follows. Given an algebra $A$ (product $ab$) denote $[a, b, c] = (ab)c - a(bc)$, the so-called associator. Santilli's Lie admissible algebras are defined by the relations

\[ [a, b, c] = 0 \]

\[ [a, b, c] + [b, c, a] + [c, a, b] = 0. \]

Note that the latter is a generalization of the Jacobi identity.


11) See p. 152 of Ref.8.

12) The association in the expression $a_1 \otimes \cdots \otimes a_n$ is arbitrary whenever not indicated (see comment after (2.9)).
13) See p. 158 of Ref. 3.

14) Actually, (2.36) does not need any special assumption to hold since we know 10) that, given a Lie algebra $\mathfrak{g}$, there always exists a Lie group whose corresponding Lie algebra is $\mathfrak{g}$.

15) Generally speaking, a Lie algebra does not need the existence of a Lie group in order to be defined. However, see footnote 14.

16) More precisely, $\exp X$ belongs to that connected part of $G$ which contains the identity element.

17) This can be shown by explicit calculation involving exponential expansions within UEMA.


19) Clearly, we can still talk about $U(1)$ generators since the UEMA contains (mod $\mathbb{Z}'$) the elements of the Lie algebra of $U(1)$.

20) Note that this relation implies a complex algebraic product. Explicitly, let us set $\lambda = 1$, $\mu = iy$. Then we have, e.g., $A \times B = AB + iyBA$ where $A$ and $B$ are (mod $\mathbb{Z}'$) Lie algebra elements. We shall comment later on the possible meaning of $y$.


29) See p. 151 of Ref. 28.

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The geodesic $\exp \{ s(X + Y) \} + \frac{\hbar}{2} \{ X, Y \}$, on the group manifold, is shifted if $\lambda - \mu \neq 1$. 

Fig. 1
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