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THE NAMBU MECHANICS AS A CLASS OF SINGULAR GENERALIZED DYNAMICAL FORMALISMS *

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ABSTRACT

In a recent work Nambu has proposed a c-number dynamical formalism which can allow an odd number n of canonical variables. Naturally associated to this new mechanics there exists an n-linear bracket whose study opens interesting possibilities. The purpose of this work is to show that besides this bracket another one which is bilinear and in fact a Lie bracket can also be associated with the Nambu mechanics. For any n , however, this bracket is singular. In a sense previously used by the present author, this result exhibits the Nambu mechanics as an interesting class of singular generalized dynamical formalisms irrespective of the number of phase space variables. Reasons are given suggesting that such singular formalisms would be, within our context, the only ones capable of describing classical analogues of generalized quantum systems.

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I. INTRODUCTION

In a recent work, Nambu (1973) proposed a c-number dynamical formalism which can allow for state spaces of odd, as well as even, dimensionality.*) According to Nambu a system whose phase space is spanned by the coordinates x^μ, $\mu=1,2,\ldots,n$, is characterized by n-1 Hamiltonians $H_1,\ldots,H_{n-1}$ which: a) are constants of motions, and b) define the dynamics through the equations:

$$\dot{x}^\mu = \frac{\partial(x^\mu, H_1, \ldots, H_{n-1})}{\partial(x^1, x^2, \ldots, x^n)} = \epsilon^{\mu\alpha_1\ldots\alpha_{n-1}} \partial_{\alpha_1} H_1 \ldots \partial_{\alpha_{n-1}} H_{n-1} \ . \tag{1.1}$$

Here, as in the following, the sum convention and the abbreviation $\partial_\lambda \equiv \frac{\partial}{\partial x^\lambda}$ have been adopted. $\epsilon^{\mu\alpha_1\ldots\alpha_{n-1}}$ denotes the totally antisymmetric Levi-Civita symbol.

The n-linear bracket

$$\{A_1, \ldots, A_n\} \equiv \frac{\partial(A_1, \ldots, A_n)}{\partial(x^1, \ldots, x^n)} \tag{1.2}$$

is naturally associated to the Nambu equations (1.1) and this bracket is proposed as the fundamental algebraic entity of the theory. According to Nambu, the quantization of his c-number theory requires finding a multilinear commutator which will define the algebraic structure of the q-number theory. The only case considered in detail in that work is that of n = 3. For this particular, and very interesting, case the quantization rule

$$[x^1, x^2, x^3] = i I \tag{1.3}$$

is postulated. Nambu suggests that the trilinear bracket be implemented by means of a commutator in the form:

$$[x^1, x^2, x^3] = [x^1, x^2]x^3 + [x^3, x^1]x^2 + [x^2, x^3]x^1 \tag{1.4}$$

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*) We are referring to one of several alternatives suggested by Nambu. This one corresponds to Eqs. (4) of Nambu's work.

Recently, interest in this theory has increased due to the following facts: a) the Nambu mechanics can describe systems with odd-dimensional phase spaces. This is not possible either within the framework of standard (Hamilton's or Dirac's) theory (Cohen & Kálnay, 1974) or by means of some of its slightly generalized versions considered in a previous work by the author (Ruggeri, 1973); and b) it has been shown by García Sucre and Kálnay (1974) that almost all the irreducible Fock representations of Green trilinear algebra are consistent with the Nambu quantization scheme (Eqs. (1.3) and (1.4)); this occurs at least for the one creation, one annihilation operator case.

The present work shows that besides the n-linear bracket (1.2), a bilinear Lie bracket can also be associated to the Nambu mechanics. This fact, proved in Sec. II, puts that formalism into a traditional and suggestive form. However, as the Lie bracket turns out to be singular, Nambu's equations are definitely not of the standard form, a fact which was mentioned previously. Finally, in Sec. III, we give a preliminary discussion of the (somewhat conjectural) relationships between c-number singular dynamical formalisms and the (eventually existing) classical analogues of generalized quantum systems.

II. THE NAMBU MECHANICS AND THE GENERALIZED DYNAMICAL FORMALISMS

As in previous works (Ruggeri, 1973, 1974), we call "generalized dynamical formalism" every set of dynamical equations of the type

$$\dot{x}^{\mu} = \Gamma^{\mu\nu} \partial_{\nu} H \; . \tag{2.1}$$

Here, a) the (analytically time parametrized) variables x^{μ} $\mu = 1,\ldots,n$, may be even or odd in number, and b) the bracket

$$\{A, B\}_{-}^{\Gamma} \equiv \Gamma^{\mu\nu} \partial_{\mu} A \, \partial_{\nu} B \tag{2.2}$$

defines a Lie product in the space of dynamical variables. This last condition is translated into the following two sets of relations (see, for instance, Ruggeri 1973):

$$\Gamma^{\mu\nu} = -\Gamma^{\nu\mu} , \tag{2.3a}$$

$$\Gamma^{\mu\nu}\partial_\nu \Gamma^{\rho\lambda} + \Gamma^{\lambda\nu}\partial_\nu \Gamma^{\mu\rho} + \Gamma^{\rho\nu}\partial_\nu \Gamma^{\lambda\mu} = 0 . \tag{2.3b}$$

The relations (2.3b) are in fact a set of $\binom{n}{3}$ differential equations which, together with (2.3a), are equivalent to the Jacobi identity. For reasons that will be obvious below, it is convenient to rewrite (2.3b) as follows. Let us introduce the totally antisymmetric Levi-Civita symbol $\epsilon_{\mu_1\mu_2\ldots\mu_{n-3}\lambda\mu\rho}$. Then it can easily be shown that we can put (2.3b) in the contracted form:

$$J_{\mu_1\mu_2\ldots\mu_{n-3}} \equiv \epsilon_{\mu_1\mu_2\ldots\mu_{n-3}\rho\lambda\mu}\,\Gamma^{\rho\nu}\partial_\nu\Gamma^{\lambda\mu} = 0 . \tag{2.4}$$

We now select (arbitrarily) one of the Nambu Hamiltonians, say H_1, and denote it simply by H. Also, we collect the remaining n-2 Hamiltonians in order to define $\Gamma^{\mu\nu}_{(N)}$ by

$$\Gamma^{\mu\nu}_{(N)} \equiv \epsilon^{\mu\nu\alpha_2\ldots\alpha_{n-1}}\,\partial_{\alpha_2}H_2 \ldots \partial_{\alpha_{n-1}}H_{n-1} . \tag{2.5}$$

In this manner the Nambu equations (1.1) can be rewritten as in Eq. (2.1):

$$\dot{x}^\mu = \Gamma^{\mu\nu}_{(N)}\;\partial_\nu H . \tag{2.6}$$

We shall now show that using Eq. (2.5) allows us to define a Lie bracket with form (2.2). For this purpose we must show that Eqs. (2.3a) and (2.4) are satisfied. The antisymmetry of $\Gamma_{(N)}$ follows immediately from the symmetry properties of $\epsilon^{\mu\nu\cdots\cdots}$. The Jacobi identity deserves a little more attention. In the present case, due to Eq. (2.5), the correspondimg J symbol of Eq. (2.4) is

$$J^{(N)}_{\mu_1\mu_2\cdots\mu_{n-3}} = \delta^{\mu\alpha_2\cdots\alpha_{n-1}}_{\mu_1\mu_2\cdots\mu_{n-3}\lambda\nu}\,\epsilon^{\lambda\nu\beta_2\beta_3\cdots\beta_{n-1}}\,\partial_{\alpha_2}H_2\cdots\partial_{\alpha_{n-1}}H_{n-1}\times \tag{2.7}$$
$$\times\,\partial_\mu\left(\partial_{\beta_2}H_2\cdots\partial_{\beta_{n-1}}H_{n-1}\right).$$

We have introduced the "generalized delta symbol" *) which satisfies

$$\delta^{\mu\alpha_2\cdots\cdots\alpha_{n-1}}_{\mu_1\mu_2\cdots\mu_{n-3}\lambda\nu} = \epsilon_{\mu_1\mu_2\cdots\mu_{n-3}\lambda\nu\rho}\,\epsilon^{\rho\mu\alpha_2\cdots\cdots\alpha_{n-1}} \tag{2.8}$$

and is defined as follows: $\delta^{\mu\alpha_2\cdots\alpha_{n-1}}_{\mu_1\mu_2\cdots\mu_{n-3}\lambda\nu}$ is equal to zero if the set $\{\mu_1\mu_2\cdots\mu_{n-3}\lambda\nu\}$ is not a permutation of the set $\{\mu\alpha_2\cdots\alpha_{n-1}\}$ and equal to $(-1)^p$ if one can pass from one set to the other by means of p transpositions. In addition, the value of the symbol is zero if two of the upper (lower) indices are equal. According to this, the only surviving terms on the right-hand side of Eq. (2.7) are those for which one of the following two kinds of possibilities are true: a) $\lambda=\mu$ and $\nu=\alpha_k$ (or $\lambda=\alpha_k$ and $\nu=\mu$) for some $k = 2,\ldots,n-1$, or b) $\lambda=\alpha_k$ and $\nu=\alpha_\ell$ for some k and some ℓ. In both cases the remaining upper indices must be a permutation of the remaining lower indices. In case a) the Jacobi identity is an automatic consequence of the antisymmetry of $\Gamma_{(N)}$. Regarding case b), besides a numerical factor, we can write the corresponding term of $J^{(N)}$ in the form **)

$$\epsilon^{\alpha_k\alpha_\ell\beta_2\cdots\beta_{n-1}}\,\partial_{\mu_1}H_2\cdots\partial_{\mu_{n-4}}H_{n-1}\Big[\sum_{i\neq k,\ell}\partial_{\beta_2}H_2\cdots\partial^2_{\mu_{n-3}\beta_i}H_i\cdots\partial_{\beta_{n-1}}H_{n-1}+$$
$$+\;\partial_{\beta_2}H_2\cdots\partial^2_{\mu_{n-3}\beta_k}H_k\cdots\partial_{\beta_{n-1}}H_{n-1}+ \tag{2.9}$$
$$+\;\partial_{\beta_2}H_2\cdots\partial^2_{\mu_{n-3}\beta_\ell}H_k\cdots\partial_{\beta_{n-1}}H_{n-1}\Big],$$

*) See, for instance, Eisenhart 1964.

**) We have chosen $\alpha_2=\mu_1,\ldots,\,\mu=\mu_{n-3}$. Any other possibility is treated analogously.

where $\partial^2_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}$. But, the three kinds of terms of this relation are zero. This can be shown at once by interchanging the role of the dummy indices α_k and β_k for the first term, of α_ℓ and β_ℓ for the second, and of α_k and β_k for the third. Then, $J^{(N)} = 0$ and the Jacobi identity is satisfied.

On the other hand, the Lie bracket just defined is always of the singular type in the sense that $\| \Gamma^{\mu\nu}_{(N)} \|$ is singular irrespective of $n > 2$; this is due to the antisymmetry of the Levi-Civita symbol since we have

$$\Gamma^{\mu\nu}_{(N)} \partial_\nu H_m = \epsilon^{\mu\nu\alpha_2\ldots\alpha_{n-1}} \partial_{\alpha_2} H_2 \ldots \partial_{\alpha_m} H_m \partial_\nu H_m \ldots \partial_{\alpha_{n-1}} H_{n-1} \tag{2.10}$$
$$= 0 \qquad , \quad m = 2, \ldots, n-1 .$$

This shows that $\| \Gamma^{\mu\nu}_{(N)} \|$ has n-2 null vectors $N^{(m)}$, $m = 2,\ldots,n-1$

$$N^{(m)}_\nu = \partial_\nu H_m , \tag{2.11}$$

so that its range is at most two. This completes the proof that the Nambu equations can be cast in the more familiar form of the equations (2.1). We stress, however, the singular character of the Lie bracket defined trough Eqs. (2.2) and (2.5), a fact which implies drastic changes in the mathematical structure of the form formalism. *)

*) These changes have not been studied in detail. However, clearly, many of the essential properties of the standard dynamical equations are heavily dependent on the non-singular character of $\| \Gamma^{\mu\nu} \|$. As an example, we mention that the singularity casts some doubt on the existence of an underlying Lagrangian for Eqs. (2.1) (cf. Ruggeri, 1973). For the Nambu equations this last fact follows easily from a recent work of Cohen and Kálnay (1974).

III. DISCUSSION

The main reason which justifies the study of c-number alternatives to the standard (Hamiltonian) one is, in our opinion, the heuristical value that such alternatives could have as classical analogues of generalized quantum systems. Generally speaking, such systems exists only on a speculative basis. Recent works, however, have given to Green parasystems a somewhat special status because, besides its eventual relevance to quarks description, a connection has been found between Bose statistics and the irreducible representations of para-Fermi algebra (Kálnay, Mac Cotrina & Kademova, 1973; Kálnay, 1974). This offers an interesting non-standard case for which the (eventually existing) classical analogues are worth studying(see in this connection, Kálnay, 1972). In a previous work (Ruggeri, 1973) the Hamilton equations were generalized, conserving their main algebraic structure, in order to look for such classical analogues. In that work, however, we restricted ourselves to the non-singular case, characterized by equations like (2.2) with a non-singular matrix $\| \Gamma^{\mu\nu} \|$. They seem to us now to be completely incapable of yielding generalized systems: the existence of coordinate transformations changing the matrix $\| \Gamma^{\mu\nu} \|$ to a constant matrix are known.*) This matrix, however, is characteristic of a Bose system.**) This seems to indicate that any non-singular generalized dynamical formalism describes a Bose system but, eventually, in unnatural coordinates. In our context, we would then be left with the singular cases as the only ones which could perhaps serve as classical analogues of Green parasystems and in fact of

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*) This one is referred to in the literature as the symplectic case.

**) In view of the above-mentioned relations between Bose algebra and para-Fermi algebra, Bose-like classical analogues can be used to construct classical para-Fermi variables. This has been done by Kálnay (1972). Our aim is basically different, being directed towards finding classical variables <u>which close a Lie algebra</u> compatible with the para-Fermi trilinear algebra. We remark that the para-Bose case has additional complications which we now prefer to avoid (see the discussion in Kálnay 1972) any generalized system. This, clearly, is compatible with the referred results of García Sucre and Kálnay (1974) relating the quantized version of the Nambu mechanics (which is singular) with the Green algebra.

Turning again to the results of the present work, it is evident that a standard (i.e. bilinear, Lie) quantization scheme can be set up. For this purpose one must decide which of the n-1 constants of motion H_m is to be selected as a Hamiltonian and consequently will be interpreted as a q-number. The remaining n-2 will retain its c-number character and collectively might well help to define the algebraic content of the theory. This is nothing but a rough conjecture about a point which deserves more investigation.

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