HYDROGEN ATOM SPECTRUM AND THE LAMB SHIFT
IN NONCOMMUTATIVE QED

M. Chaichian
High Energy Physics Division, Department of Physics, University of Helsinki
and
Helsinki Institute of Physics, FIN-00014 Helsinki, Finland,

M.M. Sheikh-Jabbari
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

A. Tureanu
High Energy Physics Division, Department of Physics, University of Helsinki
and
Helsinki Institute of Physics, FIN-00014 Helsinki, Finland.

Abstract

We have calculated the energy levels of the hydrogen atom and as well the Lamb shift within the noncommutative quantum electrodynamics theory. The results show deviations from the usual QED both on the classical and on the quantum levels. On both levels, the deviations depend on the parameter of space/space noncommutativity.

MIRAMARE – TRIESTE
October 2000

E-mail addresses: Masud.Chaichian@helsinki.fi; jabbari@ictp.trieste.it; atureanu@pcu.helsinki.fi
Recently, remotivated by string theory arguments, noncommutative spaces (Moyal plane) have been studied extensively. The noncommutative space can be realized by the coordinate operators satisfying

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \]

where \( \hat{x} \) are the coordinate operators and \( \theta_{\mu\nu} \) is the noncommutativity parameter and is of dimension of \((\text{length})^2\); for a review on the string theory side, see [1]. The action for field theories on noncommutative spaces, NCFT's, is then obtained using the Weyl-Moyal correspondence [2–4], according to which, in order to find the noncommutative action, the usual product of fields should be replaced by the star-product:

\[ (f \ast g)(x) = \exp \left( \frac{i}{2} \theta_{\mu\nu} \partial_{x_\mu} \partial_{y_\nu} \right) f(x)g(y)|_{x=y}, \]

where \( f \) and \( g \) are two arbitrary infinitely differentiable functions on \( R^{3+1} \). Performing explicit loop calculations, for \( \theta^{0i} = 0 \) cases (noncommutative space), it has been shown that noncommutative \( \phi^4 \) theory up to two loops [3,5] and NCQED up to one loop [6,4,7], are renormalizable. For noncommutative space-time (\( \theta^{0i} \neq 0 \)) it has been shown that the theory is not unitary and hence, as a field theory, it is not appealing [8].

Apart from the field theory interests which are more academic, we are more interested in some possible phenomenological consequences of noncommutativity in space. Some of those results, all from the field theory point of view, have been addressed in [4,11,12]. However, perhaps a better starting point is to study quantum mechanics (QM) on such noncommutative spaces. To develop the NCQM formulation we need to introduce a Hamiltonian which governs the time evolution of the system. We should also specify the phase space and, of course, the Hilbert space on which these operators act. As for the phase space, inferred from the string theory [9,10], we choose

\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \]
\[ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \]
\[ [\hat{p}_i, \hat{p}_j] = 0. \]  

The Hilbert space can consistently be taken to be exactly the same as the Hilbert space of the corresponding commutative system. This assumption for the Hilbert space is directly induced from the non-relativistic limit of the related NCFT, and one can really argue that it satisfies all the needed properties of a physical Hilbert space. The only non-trivial part of such a formulation is to give the Hamiltonian. Once we have done it, the dynamical equation for the state \(|\psi\rangle\) is the usual Schrödinger equation, i.e.

\[ H|\psi\rangle = i\hbar \frac{\partial}{\partial \Re} |\psi\rangle. \]

In this letter we focus on the hydrogen atom and, using the non-relativistic limit of NCQED results, we propose the Hamiltonian describing the NC H-atom. Given the Hamiltonian and assuming that the noncommutativity parameter \( (\theta_{ij}) \) is small, we study the spectrum of H-atom. We show that because of noncommutativity, even at field theory tree level, we have some corrections to the Lamb shift \((2P \rightarrow 2S \) transition). Since the noncommutativity in space violates rotational symmetry, our Lamb shift corrections have a preferred direction and hence we call them “polarized Lamb shift”. We also consider further
corrections to Lamb shift originating from the loop contributions in NCQED. In this way we will find some upper bound for $\theta$. In addition, we study the Stark and Zeeman effects for the noncommutative H-atom.

II. FORMULATION OF THE NONCOMMUTATIVE HAMILTONIAN

To start with, we propose the following Hamiltonian for the noncommutative H-atom. Of course, we shall verify our proposal by a NCQED calculation:

$$H = \frac{\hat{p} \cdot \hat{p}}{2m} + V(\hat{x}) ,$$

(2.1)

where the Coulomb potential in terms of the noncommutative coordinates $\hat{x}$ is:

$$V(r) = -\frac{Ze^2}{\sqrt{\hat{x} \cdot \hat{x}}} ,$$

(2.2)

with $\hat{p}$ and $\hat{x}$ satisfying (1.3).

Now, we note that there is a new coordinate system,

$$x_i = \hat{x}_i + \frac{1}{2\hbar} \theta_{ij} \hat{p}_j ,$$

$$p_i = \hat{p}_i ,$$

(2.3)

where the new variables satisfy the usual canonical commutation relations:

$$[x_i, x_j] = 0 ,$$

$$[x_i, p_j] = i\hbar \delta_{ij} ,$$

$$[p_i, p_j] = 0 .$$

(2.4)

So, if in the Hamiltonian we change the variables $\hat{x}_i$, $\hat{p}_i$ to $x_i$, $p_i$, the Coulomb potential becomes:

$$V(r) = -\frac{Ze^2}{\sqrt{(x_i - \theta_{ij} p_j/2\hbar)(x_i - \theta_{ik} p_k/2\hbar)}}$$

$$= -\frac{Ze^2}{r} - \frac{Ze^2}{2\hbar r^3} \theta_{ij} p_j + O(\theta^2)$$

$$= -\frac{Ze^2}{r} - \frac{Ze^2 L \cdot \theta}{4\hbar r^3} + O(\theta^2) ,$$

(2.5)

where $\theta_{ij} = \frac{1}{2} \epsilon_{ijk} \theta_k$, $L = r \times p$.

As $(r \times p) \cdot \theta = -r \cdot (\theta \times p)$, it follows that the Coulomb potential can also be written as

$$V(r) = -\frac{Ze^2}{r} - \frac{e}{4\hbar} (\theta \times p) \cdot \left( -\frac{Ze^2}{r^3} \right) + O(\theta^2) .$$

(2.6)

The other higher order terms, besides being higher powers in $\theta$ which in its own turn is very small, are also higher powers in momenta.

Our proposal for Hamiltonian can be justified from field theory calculations. The electron-photon vertex function at tree level in NCQED is [4]:

$$\Gamma_\mu = e^{\frac{i}{\hbar} \gamma_\mu \cdot p \times p} \gamma_\mu$$

$$= e^{-\frac{i}{\hbar} \gamma_\mu \cdot q} \gamma_\mu ,$$

(2.7)
where \( p \) and \( p' \) are the in-coming and out-going electron momenta, respectively, \( q_\mu \) is the photon momentum: \( p' - p = q \) and

\[
p \times p' = p_i \theta^{ij} p'_j, \quad \theta^{ij} = \theta^{ij} q_j.
\]

Expanding the exponential in powers of \( \theta \) and keeping only the first two terms, it appears that the second term will give rise to an electric dipole moment [13], which couples to an external electric field \( E \) as \(-\langle P_i \rangle \cdot E\), where

\[
\langle P_i \rangle = \frac{1}{2\hbar} c \hat{p}_i = \frac{1}{2\hbar} c \theta^{ij} p_j.
\]

This electric dipole moment, as we will see, changes the usual Lamb shift. Actually one can go further and prove that the potential (2.5), for all orders in \( \theta \), is expected from the NCQED starting from (2.7). This can be done noting that \( f(x_i + \epsilon_\theta) = e^{\epsilon_\theta \frac{\partial}{\partial x_i}} f(x) \).

Our proposal for the NC H-atom Hamiltonian can be generalized to other systems, i.e. taking the usual Hamiltonian but now being a function of noncommutative coordinates (like (2.1)). However, our discussion based on NCQED is only applicable when we deal with the “electro-magnetic” interaction. In other words, at field theory tree level and in the non-relativistic limit, the noncommutativity of space is probed through the electric dipole moment of particles, whether fermions or bosons.

In our formulation for NCQM, one can still use the usual definition for the probability density, \(|\psi|^2\). However, one should be aware that there is no coordinate basis in this case. In our approach, since the noncommutativity parameter, if it is non-zero, should be very small compared to the length scales of the system, one can always treat the noncommutative effects as some perturbations of the commutative counter-part and hence, up to first order in \( \theta \), we can use the usual wave functions and probabilities.

III. “CLASSICAL” SPECTRUM FOR HYDROGEN ATOM IN NC THEORY

Using the usual perturbation theory, the leading corrections to the energy levels due to noncommutativity, i.e. first order perturbation and in field theory tree level, are:

\[
\Delta E_{\text{NC-atom}}^{\text{H-atom}} = -\langle n^l m^l | \frac{Z e^2}{4\hbar} \frac{L \cdot \theta}{r^3} | nlm \rangle.
\]

We note that the above expression is very similar to that of the spin-orbit coupling, where \( \frac{\theta}{\hbar} \) is now replacing the spin, \( \frac{S}{\hbar} \), with \( \lambda_e \) being the electron Compton wave length.

If we put \( \theta = \theta \) and the rest of the \( \theta \)-components to zero (which can be done by a rotation or a redefinition of coordinates), then \( L \cdot \theta = L_z \theta \) and, taking into account the fact that \( L_z | nlm \rangle = m \hbar | nlm \rangle \), the energy level shift given by (3.1) becomes [14]:

\[
\Delta E_{\text{NC}} = -\frac{m e^2}{4} \left( Z e \right) \frac{\theta}{\lambda_e^2} \frac{m}{n^2(l + \frac{1}{2})(l + 1)} \delta_{l,l'} \delta_{m,m'}.
\]

One should note that for \( l = 0 \) (S-orbit) (3.2) is not valid, and this is more or less what also happens for the usual spin-orbit coupling. The reason is that, in order to find \( \langle \frac{1}{r^3} \rangle \), one should integrate over the wave
functions from \( r = 0 \); on the other hand, since the wave function at \( r = 0 \) is not zero for the \( l = 0 \) case, the approximation (dropping the terms higher order in \( \theta \)) is not valid any more. However, this situation is regularized by changing the lower limit of integration by \( \sqrt{\theta} \).

The case of our interest, the \( 2P \rightarrow 2S \) transition (Lamb shift), differs from the usual commutative case in which the shift depends only on the \( l \) quantum number and all the corrections are due to the field theory loop effects. The Lamb shift for the noncommutative H-atom, besides the usual loop effects, depends on the \( m \) quantum number and is there, even in the field theory tree level. Hence we call it \textit{polarized Lamb shift}. More precisely, there are some \textit{new} transition channels which are opened because of noncommutativity: \( 2P^{-1} \rightarrow 2P^0 \) and \( 2P^0 \rightarrow 2P^{+1} \). The usual Lamb shift, \( 2P \rightarrow 2S \), is now split into two parts, \( 2P^{+1} \rightarrow 2S \) and \( 2P^{-1} \rightarrow 2S \), which means that the noncommutativity effects increase the widths and splits of the Lamb shift line by a factor proportional to \( \theta \).

### IV. ONE-LOOP CORRECTIONS

In the usual commutative theory, the Lamb shift is believed to come from \textit{loop corrections} to QED. In the usual case, both vertex corrections, in particular, the \( g - 2 \) factor in the spin-orbit coupling, and the corrections to photon propagator [15] are responsible for the Lamb shift.

#### A. Noncommutative one loop vertex corrections

According to NCQED one loop results, the electric and magnetic dipole moments of the electron, as a Dirac particle, are [4]:

\[
\begin{align*}
\langle \vec{p} \rangle &= \frac{-e}{2me} \left( g^S + \frac{\alpha e}{3\pi} \hbar \frac{\vec{\theta}}{\lambda^2} \right), \quad g = 2 + \frac{\alpha}{\pi}, \\
\langle \vec{P} \rangle &= -\frac{e}{\hbar} (\vec{\theta} \times \vec{p})(1 + \frac{3\alpha e}{\pi}).
\end{align*}
\]

Hence, the \textit{noncommutative} one loop correction to the potential (2.5), originating from \textit{vertex} corrections up to the first order in \( \theta \), is:

\[
V_{\text{NC vertex}}^{1 \text{Loop}} = -\frac{Ze^2}{4\pi} \gamma_E \alpha (3 - \frac{2}{3}) \frac{\vec{L} \cdot \vec{\theta}}{\hbar \gamma^3}.
\]

#### B. Noncommutative one loop photon propagator corrections

The photon propagator at one loop in the NCQED, for small \( q, \vec{q} \) is given by [6]:

\[
\Pi^{\mu\nu}(q) = \frac{e^2}{16\pi^2} \left[ \frac{10}{3} (g^{\mu\nu} q^2 - q^\mu q^\nu) \left( \ln(q^2\vec{q}^2) + \frac{2}{25} \frac{q^2}{m^2} \right) \\
+ 32 \frac{\vec{q} \cdot \vec{q}'}{q^4} - \frac{4}{3} \frac{q^2 q^2 q^2 q^2}{q^4} \right],
\]

where the term proportional to \( \frac{2}{25} \frac{q^2}{m^2} \) is the fermionic loop contribution which, because of the cancellation in phase factors coming from noncommutativity, is the same as the usual QED result. From (4.3), by
taking only the part of the propagator corresponding to time-like photons and reintroducing $h, c$ factors, in the units where the Coulomb potential is $-\frac{Z e^2}{r}$, we obtain

$$V_{\text{prop.}}^{1\text{Loop}}(r) = -Z e^2 \alpha \frac{10}{3h} \int d^3 q \frac{1}{q^2} e^{-i\mathbf{q} \cdot \mathbf{r}/\hbar} \times \left( \ln \left( \frac{q^2}{\hbar^4} \right) + \frac{2}{25} \frac{q^2}{m^2} \right).$$  (4.4)

The second term in the integral yields the usual $\delta^3(r)$ type correction to the Coulomb potential. To work out the integral in the first term which is $\theta$ dependent, let us assume that only $\theta_{12} = \theta$ is non-zero. If we denote the integral by $I(r, \theta)$, then $\frac{d}{d \theta} = \frac{1}{2\pi r^2}$ and thus $I(\theta, r) = \frac{1}{2\pi r} \ln (\theta \Lambda^2)$, where $\Lambda$ is a cut-off. This can be understood noting that, because of IR/UV mixing [4,6], the Fourier transformation and also (4.3), are valid for $\frac{1}{\Lambda} \lesssim q \lesssim \Lambda$. Putting all these results together, we have

$$V_{\text{prop.}}^{1\text{Loop}}(r) = \frac{Z e^2}{2\pi} \frac{10}{3} \ln (\theta \Lambda^2) - Z e^2 \frac{4\alpha}{15} \chi^2 \delta^3(r).$$  (4.5)

The first term being proportional to $\frac{1}{r}$, can be understood as the normalization of charge at one loop level [6]; however, to find the physical value of $\alpha$ (NCQED coupling), one should study the Thomson limit of Compton scattering [16] for the noncommutative case [17]. Summing up all the one loop contributions to Lamb shift due to noncommutativity, (4.2), (4.5), we get

$$\Delta E_{NC}^{1\text{Loop}} = -\frac{1}{2\pi} m_e c^3 (Z\alpha)^2 \left[ \frac{5\alpha}{3} \ln (\theta \Lambda^2) \frac{1}{n^2} - \frac{(Z\alpha)^2}{2} \frac{\theta}{\lambda_c^2} \right] \times \gamma E \alpha \left( \frac{2}{3} n^2 \right) \frac{m}{n^2(l + \frac{1}{2})(l + 1)}.$$  (4.6)

One can use the data on the Lamb shift to impose some bounds on the value of the noncommutativity parameter, $\theta$. Of course, to do it, we only need to consider the classical (tree level) results, (3.2). Comparing these results, the contribution of (3.2) should be of the order of $10^{-6} - 10^{-7}$ smaller than the usual one loop result and hence,

$$\frac{\theta}{\lambda_c^2} \lesssim 10^{-7} \alpha \quad \text{or} \quad \theta \lesssim (10^4 \text{ GeV})^{-2}.$$

This bound is indeed not a strong one, and one would need some more precise experiments or data. Among other processes, the $e^+e^-$ scattering data can provide a better bound on $\theta$ [17].

V. NONCOMMUTATIVE STARK AND ZEEMAN EFFECTS

Stark effect

The potential energy of the atomic electron in an external electric field oriented along the $z$-axis is given, at tree level, by

$$V_{\text{Stark}} = eEz + \frac{e}{4\hbar} (\theta \times p) \cdot E.$$  (5.1)

(neglecting the motion of the proton).
The change in the hydrogen atom energy levels due to noncommutativity (the second term in (5.1)) is:

$$\Delta E^{NC}_{\text{Stark}} = \langle nl'm'| \frac{e}{4\hbar} (\theta \times p) \cdot E |nlm \rangle ,$$  \hspace{1cm} (5.2)

Taking into account the fact that  $p_i = \frac{m}{\hbar} [x_i, H_0]$, where $H_0$ is the unperturbed Hamiltonian, so that $H_0|nlm \rangle = E_n |nlm \rangle$, the correction to the energy levels becomes:

$$\Delta E_{\text{Stark}}^{NC} = \frac{em}{4i\hbar^2} (\theta \times E)_i \langle nl'm'| [x_i, H_0] |nlm \rangle = 0 ,$$  \hspace{1cm} (5.3)

meaning that, at tree level, the contribution to the Stark effect due to noncommutativity is zero. We also note that, adding the one loop corrections to electric dipole moment (4.1), the above result will not be changed.

**Zeeman effect**

The new parts which are added to the usual potential energy of the atom in a magnetic field, due to noncommutativity, are:

$$V_{\text{NC Zeeman}} = \frac{e}{2mc} \frac{\alpha \gamma E m_e^2}{3\pi\hbar} (1 - f \frac{m_p}{m_e} ) \vec{\theta} \cdot \vec{B} .$$  \hspace{1cm} (5.4)

where $f$ is a form factor of the order of unity, as the proton is not point-like. As a result, the noncommutative contribution to the Zeeman effect in the first order of perturbation theory is:

$$\Delta E_{\text{Zeeman}}^{NC} = \frac{1}{6\pi\hbar} e \alpha \gamma E m_e (1 - f \frac{m_p}{m_e} ) \vec{\theta} \cdot \vec{B} .$$  \hspace{1cm} (5.5)

**VI. CONCLUSION**

We have presented the results on the classical Coulomb potential within the formulated noncommutative quantum mechanics for the hydrogen atom and have obtained the corrections to the Lamb shift using the NCQED. If there exists any noncommutativity of space-time in nature, as it seems to emerge from different theories and arguments, its implications should appear in physical systems such as the one treated in this letter. A detailed analysis of the results obtained here, together with the treatment of other fundamental and precisely measured physical processes, will be given in a further communication [17].

**Acknowledgments**

We are grateful to A. Demichev, D. Demir, P. Prešnajder and especially to C. Montonen for many useful discussions and comments. This work was partially supported by the Academy of Finland, under the Project No. 163394. The work of M.M. Sh.-J. was partly supported by the EC contract no. ERBFMRX-CT 96-0090.
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